THE FUNDAMENTAL GROUP AND BROUWER'S FIXED POINT THEOREM

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ABSTRACT. The fundamental group is an invariant of topological spaces that measures the contractibility of loops. This project studies the fundamental group, its basic properties, some elementary computations, and a resulting theorem: the Fixed Point Theorem of Brouwer, which states that every continuous function mapping the disk to itself has a fixed point.

1. INTRODUCTION

The fundamental group ties together the ideas of topological spaces, homotopy, and group theory. The fundamental group can help answer the question of whether two topological spaces are not homeomorphic. A resulting theorem from studying the fundamental group is the Fixed Point Theorem of Brouwer, which has applications in areas such as economics, game theory, and other fields of math.

Given a topological space, a *path* can be thought of as a continuous way to move from one point to another point. If the path starts and ends at the same point, the path is called a *loop*.

FIGURE 1. Three different paths and three different loops
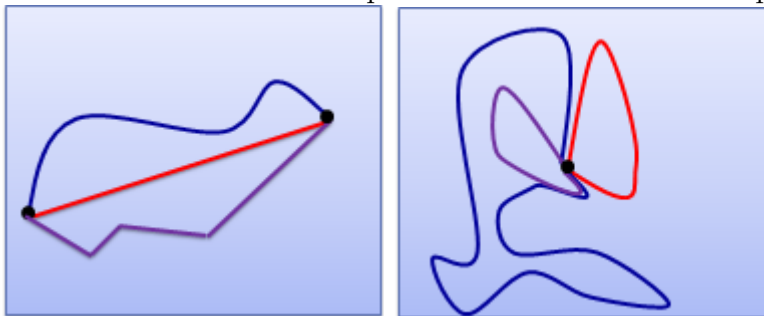
Two paths are *path homotopic* if one can be continuously deformed into the other. In Figure 2, the first picture shows two paths, and the second picture shows how the two paths can be continuously deformed into one another. Hence, the two paths are path homotopic. If two paths are path homotopic, they can be considered the same in some sense.

Concatenation of two paths can be defined on two paths if the end point of the first path is the initial point of the second path. As pictured in Figure 3, the resulting path is the path obtained by first traveling on the first path with double the speed followed by traveling along the second path with double the speed.

Take a topological space X and choose a point $x_0 \in X$. Let $\pi_1(X, x_0)$ be the set of path homotopy classes of loops based at x_0. With the operation of concatenation, it can be shown
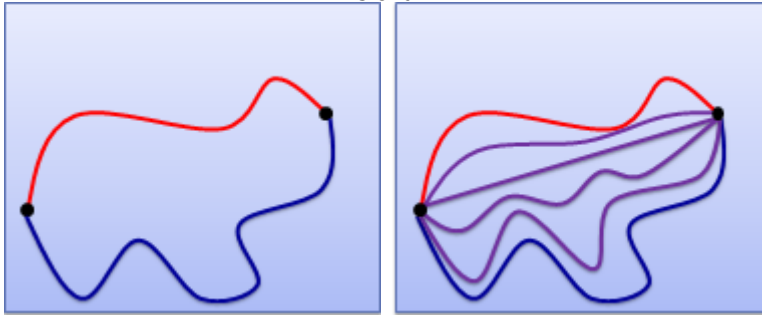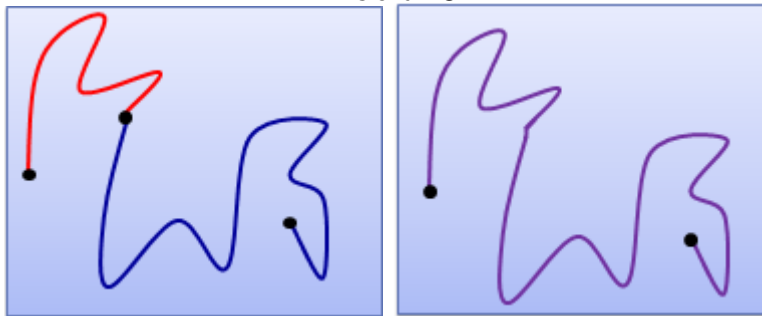
FIGURE 2

FIGURE 3

$\pi_1(X, x_0)$ is a group called the *fundamental group*[1] *of X relative to the base point x_0*. The identity of this group is the path homotopy class containing the constant path x_0. The inverse of an element is obtained by travelling a representative loop in reverse.

Note that choosing a different basepoint may result in a nonisomorphic fundamental group. However, if two points are in the same path component, then each point will produce the same fundamental group up to isomorphism.

2. FUNDAMENTAL GROUP EXAMPLES

We now give formal definitions to help us look at specific examples. In the following discussion, let $I = [0, 1]$.

Definition 2.1. Let X be a topological space. Let $a, b \in X$. A *path* from a to b is a continuous function $f : [0, 1] \to X$ such that $f(0) = a$ and $f(1) = b$.

Definition 2.2.
Let f and f' be two paths in a topological space X such that $f(0) = f'(0) = x_0$, $f(1) = f'(1) = x_1$. The path f is said to be *path homotopic* to f' if there exists a continuous map

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[1]A group is a set G with an associative binary operation $* : G \times G \to G$ such that there is an identity element and every element has an inverse with respect to $*$

$F : I \times I \to Y$ such that for all $s, t \in I$

$$F(0, t) = x_0$$
$$F(1, t) = x_1$$
$$F(s, 0) = f(s)$$
$$F(s, 1) = f'(s)$$

We call F a *path homotopy* between f and f'. In this case, we write $f \simeq_p f'$. It can easily be shown that $\simeq_p$ is an equivalence relation on X. The equivalence classes under $\simeq_p$ are called *path homotopy classes* and are the elements of the fundamental group.

Lemma 2.3. *If $A \subseteq \mathbb{R}^n$ is convex and $a_0 \in A$, then $\pi_1(A, a_0)$ is the trivial group.*

Proof: Let f be any loop based at a_0 in A. Let c_{a_0} be the constant loop based at a_0. We will show $f \simeq_p c_{a_0}$.

Consider $F : I \times I \to A$ such that

$$F(s, t) = (1 - t)f(s) + ta_0$$

Note that the image of F is contained in A since A is convex, so F is well-defined. Also, since $(1 - t)$, ta_0, and f are continuous, F is continuous since F is made up of the product and sum of continuous functions. Moreover,

$$F(0, t) = (1 - t)f(0) + ta_0 = (1 - t)a_0 + ta_0 = a_0$$
$$F(1, t) = (1 - t)f(1) + ta_0 = (1 - t)a_0 + ta_0 = a_0$$
$$F(s, 0) = f(s)$$
$$F(s, 1) = a_0 = c_{a_0}.$$

Hence, F is a path homotopy between f and c_{a_0}. Therefore, $\pi_1(A, a_0) = \{[c_{a_0}]\} \Rightarrow \pi_1(A, a_0)$ is the trivial group. $\square$

Since a convex space has exactly one path component, for any $a_0, a_1 \in A$, $\pi_1(A, a_0) \cong \pi_1(A, a_1)$.

Example 2.4. We immediately get the result that $\pi_1(D^2, x_0) \cong (\{0\}, +)$ since D^2 is convex. Hence, the fundamental group of the disk is the trivial group.

Example 2.5. $\pi_1(S^1, x_0) \cong (\mathbb{Z}, +)$. There is a correspondence between traveling around the circle and the integers. For instance, traveling around the circle once counterclockwise is like the number 1. Similarly, traveling around the circle once clockwise is like the number -1. Hence, traveling around the circle once counterclockwise followed by once clockwise can be thought of as standing still, which is the same as the number $-1 + 1 = 0$. Since the fundamental group of the circle is isomorphic to the integers with addition, we know that the fundamental group of the circle is cylic. In fact, the path homotopy class containing the path that travels around the circle once counterclockwise and the path homotopy class containing the path that travels around the circle once clockwise are the two generators. For a formal proof of this result, see Munkres [2, see pages 345–346].

Example 2.6. $\pi_1(T, x_0) \cong (\mathbb{Z} \times \mathbb{Z}, +)$. The two generators of the fundamental group of the torus in red and blue are pictured in Figure 4.

FIGURE 4
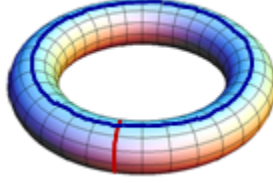
3. DISTINGUISHING TOPOLOGICAL SPACES WITH THE FUNDAMENTAL GROUP

We can use the fundamental group to help distinguish between different topological spaces. Let $\mathbf{Top}_*$ be the category of pointed topological spaces with basepoint preserving continuous maps as arrows and let $\mathbf{Grp}$ be the category of groups with homomorphisms as arrows.

Remark 3.1. $\pi_1 : \mathbf{Top}_* \to \mathbf{Grp}$ *is a functor.*

Let (X, x_0) and (Y, y_0) be pointed topological spaces. Let $h : (X, x_0) \to (Y, y_0)$ be a continuous map. We will define $\pi_1(h) := h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ where $h_*([f]) = [h \circ f]$. The map h_* is called the *homomorphism induced by h*.

Proof: Let (X, x_0), (Y, y_0), (Z, z_0) be pointed topological spaces and let $g : (X, x_0) \to (Y, y_0)$, $h : (Y, y_0) \to (Z, z_0)$ be continuous maps. It is clear what the object function is.

Consider the arrow function as described above which takes a continuous map h to the homomorphism induced by h.

We must first show $\pi_1(h \circ g) = \pi_1(h) \circ \pi_1(g)$. We will show this equality by showing the two functions agree pointwise. Let $[f] \in \pi_1(X, x_0)$. Then,

$$
\begin{aligned}
\pi_1(h \circ g)([f]) &= (h \circ g)_*([f]) \\
&= [(h \circ g) \circ f] \\
&= [h \circ (g \circ f)] \\
&= h_*([g \circ f]) \\
&= h_*(g_*([f])) \\
&= (h_* \circ g_*)([f]) \\
&= (\pi_1(h) \circ \pi_1(g))([f])
\end{aligned}
$$

Also,

$$
\begin{aligned}
\pi_1(1_{(X,x_0)})([f]) &= 1_{(X,x_0)*}([f]) \\
&= [1_{(X,x_0)} \circ f] \\
&= [f] \\
&= 1_{\pi_1(X,x_0)}([f]).
\end{aligned}
$$

Hence, π_1 is a functor. $\square$

Corollary 3.2. π_1 *takes homeomorphisms to isomorphisms.*

With this corollary, we see that if two spaces are homeomorphic, they must have the same fundamental group up to isomorphism. In the above examples, each space has only one path component. Therefore, choosing a different base point will result in the same fundamental

group up to isomorphism. Because each topological space in the previous examples has a different fundamental group from the others, none of these spaces are homeomorphic to one another.

We have claimed if two spaces are homeomorphic, they must have the same fundamental group. However, the converse is not true. For instance, the fundamental group of S^2 is the trivial group and the fundamental group of $\mathbb{R}$ is also the trivial group. However, $\mathbb{R}$ is not homeomorphic to S^2 since the latter is compact and the former is not.

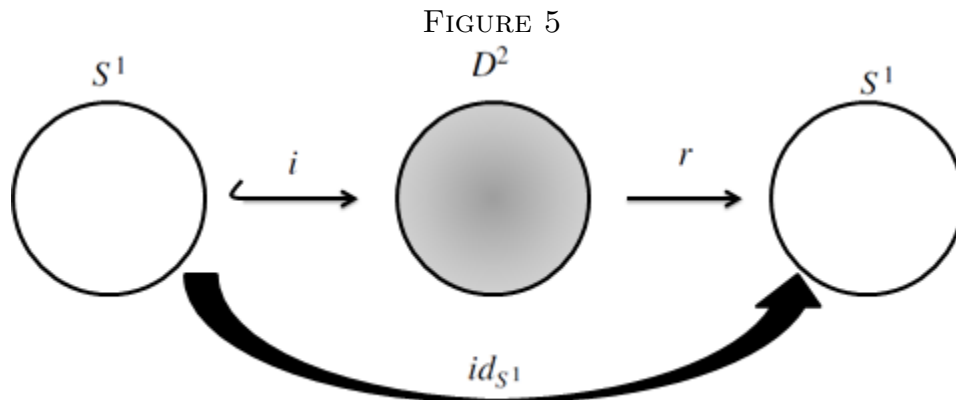## 4. THE FIXED POINT THEOREM OF BROUWER

Theorem 4.1 (The Fixed Point Theorem of Brouwer). *Every continuous function mapping the disk to itself has a fixed point.*

Before proving the Fixed Point Theorem of Brouwer, we will first prove a useful lemma, which uses the fact that $\pi_1 : \mathbf{Top}_* \to \mathbf{Grp}$ is a functor.

Definition 4.2. Let X be a topological space $A \subseteq X$ is a retraction if there exists a continuous function $f : X \to A$ such that $f|_A = id_A$.

Lemma 4.3. *There is no retraction from S^1 to D^2.*

Proof: Proceeding by contradiction, suppose there exists a retraction r from S^1 to D^2. Let i be the inclusion from S^1 to D^2. Note, $r \circ i = id_{S^1}$.

FIGURE 5

Since π_1 is a functor and by Example 2.4 and Example 2.5, we get Figure 6.

This is a contradiction since this would mean there exists a surjection from a single point set to all of $\mathbb{Z}$. □

Now, we are ready to prove Brouwer's Fixed Point Theorem using the idea of May [1, see page 10].

Proof: Let $f : D^2 \to D^2$ be continuous. Proceeding by contradiction, suppose there is no fixed point of f. In other words, for every $x \in D^2$, $f(x) \neq x$.

Define a new function $r : D^2 \to D^2$. For $x \in D^2$, let $r(x)$ be the tip of the ray on S^1 that begins at $f(x)$ and passes through x. Note that r is well-defined since $f(x) \neq x$ means that $r(x) \in S^1$ for all $x \in D^2$. Since f is continuous it follows that r is continuous. The map $r(x)$ is pictured in Figure 7.
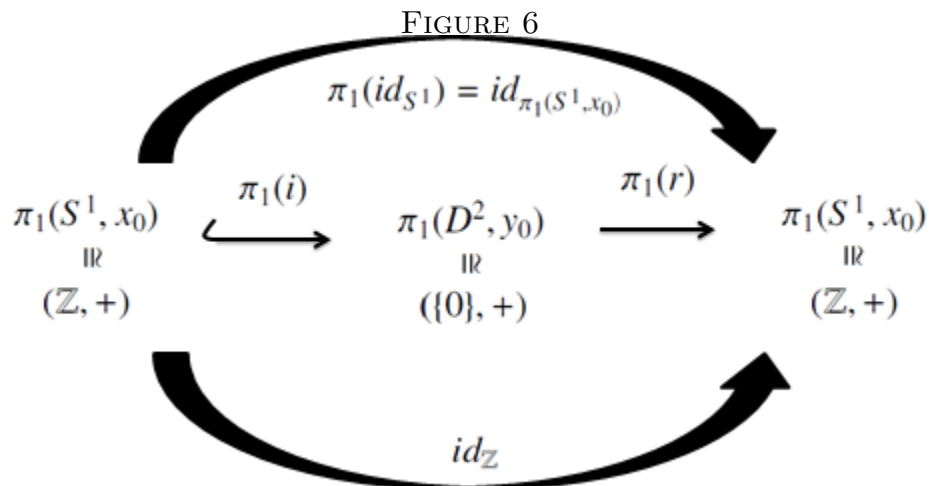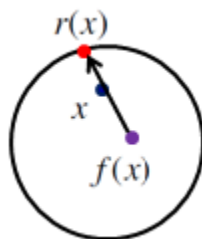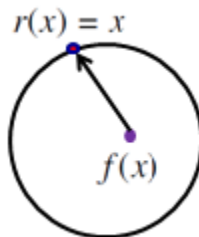
FIGURE 6

$$\pi_1(id_{S^1}) = id_{\pi_1(S^1, x_0)}$$

$$\pi_1(S^1, x_0) \quad \xrightarrow{\pi_1(i)} \quad \pi_1(D^2, y_0) \quad \xrightarrow{\pi_1(r)} \quad \pi_1(S^1, x_0)$$

$$\mathbb{R} \qquad\qquad\qquad \mathbb{R} \qquad\qquad\qquad \mathbb{R}$$

$$(\mathbb{Z}, +) \qquad\qquad (\{0\}, +) \qquad\qquad (\mathbb{Z}, +)$$

$$id_{\mathbb{Z}}$$

FIGURE 7

$r(x)$

x

$f(x)$

But what happens when x lies on the edge of the circle? We have that $r(x) = x$ when $x \in S^1$. See Figure 8.

FIGURE 8

$r(x) = x$

$f(x)$

Hence, $r|_{S^1} = id_{S^1}$. So r is actually a retraction of S^1 to D^2. This is a contradiction by the previous lemma since no retraction of the circle to the disk can exist. Therefore, every continuous function mapping the disk to itself has a fixed point. $\qquad\square$

REFERENCES

[1] J. P. May. *A concise course in algebraic topology.* Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1999.

[2] James R. Munkres. *Topology: a first course.* Prentice-Hall Inc., Englewood Cliffs, N.J., 1975.