The Fundamental Group and Brouwer's Fixed Point Theorem

Amanda Bower, Faculty Sponsor: Thomas Fiore

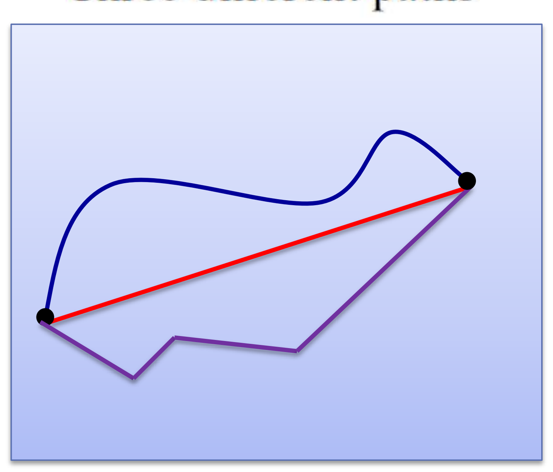
DEARBORN
College of Arts, Sciences, and Letters

Introduction

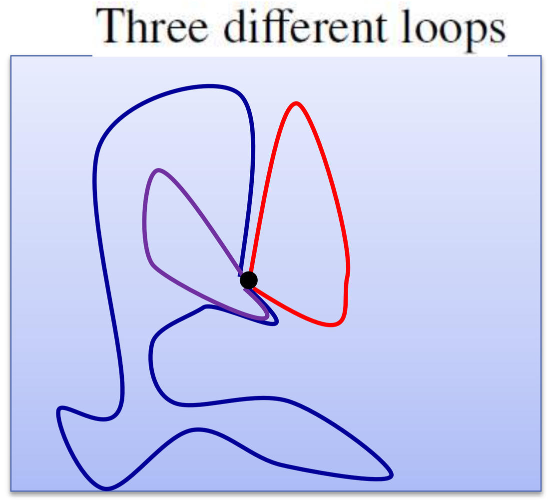
The fundamental group ties together the ideas of topological spaces, homotopy, and group theory. The fundamental group can help answer the question of whether two topological spaces are not homeomorphic. A resulting theorem from studying the fundamental group is the Fixed Point Theorem of Brouwer, which has applications in areas such as economics, game theory, and other fields of math.

Given a topological space, a *path* can be thought of as a continuous way to move from one point to another point. If the path starts and ends at the same point, the path is called a *loop*.

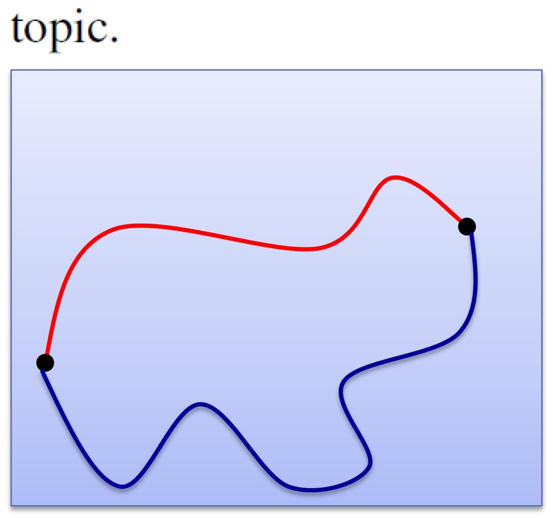
Three different paths

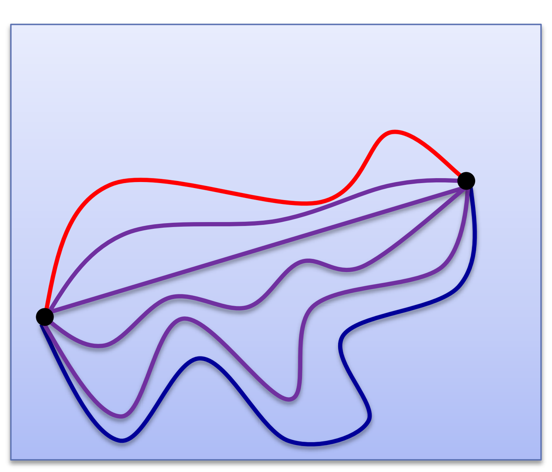


Three different loops



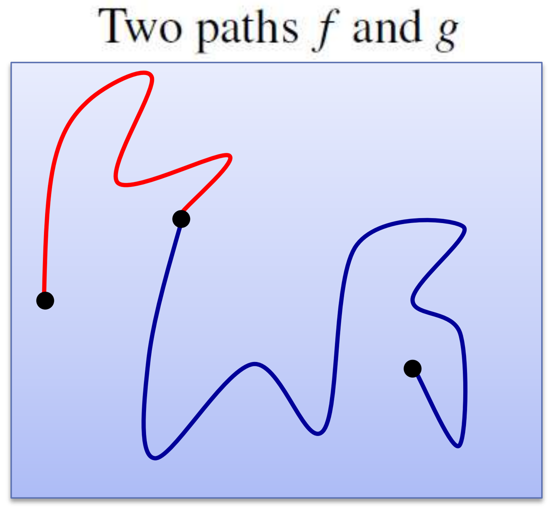
Two paths are *path homotopic* if one can be continuously deformed into the other. The first picture shows two paths, and the second picture shows how the two paths can be continuously deformed into one another. Hence, the two paths are path homotopic.



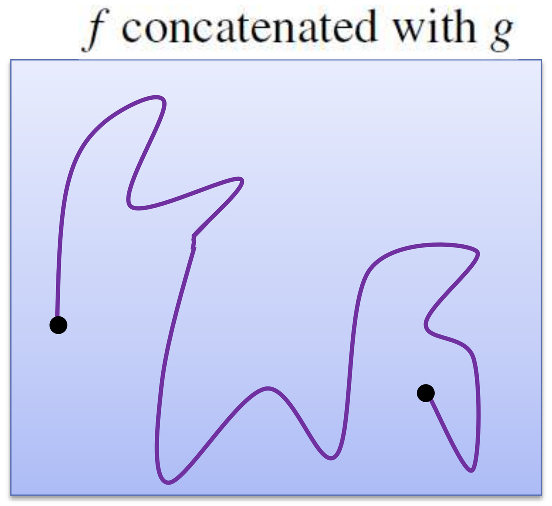


Concatenation of two paths can be defined on two paths if the end point of the first path is the initial point of the second path. The resulting path is the path obtained by first traveling on the first path with double the speed followed by traveling along the second path with double the speed.

Two paths f and g



f concatenated with g



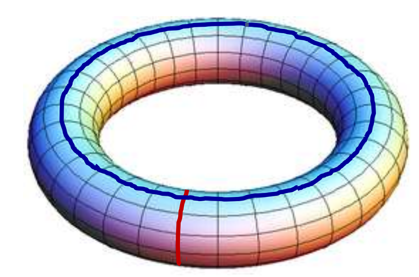
Take a topological space X and choose a point $x_0 \in X$. Let $\pi_1(X, x_0)$ be the set of path homotopy classes of loops based at x_0. With the operation of concatenation, it can be shown $\pi_1(X, x_0)$ is a group called the *fundamental group* of X relative to the *base point* x_0. The inverse of an element is obtained by travelling a representative loop in reverse.

Fundamental Group Examples

Example 1: $\pi_1(D^2, x_0) \cong (\{0\}, +)$. In D^2, all loops can be continuously deformed into a single point. In other words, all loops based at x_0 are homotopic to the constant path x_0. Hence, the disk's fundamental group is the trivial group.

Example 2: $\pi_1(S^1, x_0) \cong (\mathbb{Z}, +)$. There is a correspondence between traveling around the circle and the integers. For instance, traveling around the circle once counterclockwise is like the number 1. Similiarly, traveling around the circle once clockwise is like the number -1. Hence, traveling around the circle once counterclockwise followed by once clockwise can be thought of as standing still, which is the same as the number $-1 + 1 = 0$.

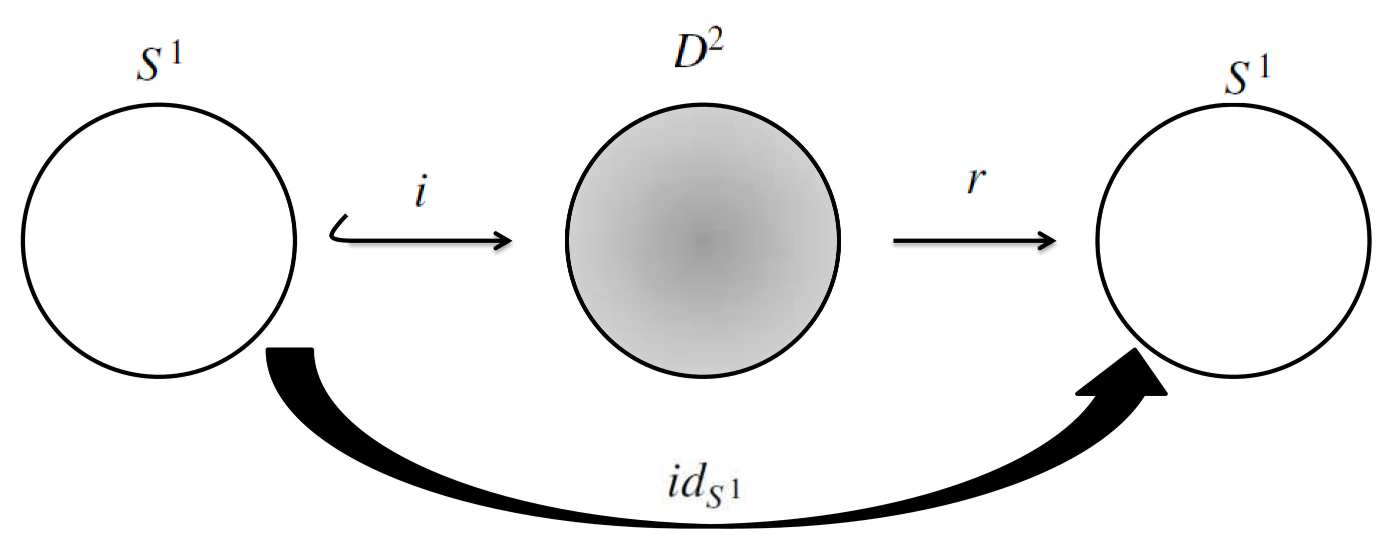
Example 3: $\pi_1(T, x_0) \cong (\mathbb{Z} \times \mathbb{Z}, +)$. The two generators of the fundamental group of the tourus in red and blue are pictured below:



Note that all three of these topological spaces have different fundamental groups. Hence, each of these spaces is topologically different from the other two spaces.

Lemma: There is no retraction from S¹ to D².

Proceeding by contradiction, suppose there exists a retraction r from S^1 to D^2. Let i be the inclusion from S^1 to D^2.



Since π_1 is a functor,

$$\begin{tikzcd}
\pi_1(S^1, x_0) \arrow[r, "\pi_1(i)"] \arrow[rr, bend left, "\pi_1(id_{S^1}) = id_{\pi_1(S^1, x_0)}"] & \pi_1(D^2, y_0) \arrow[r, "\pi_1(r)"] & \pi_1(S^1, x_0) \\
(\mathbb{Z}, +) \arrow[u, phantom, "\cong"] \arrow[rr, bend right, "id_{\mathbb{Z}}"'] & (\{0\}, +) \arrow[u, phantom, "\cong"] & (\mathbb{Z}, +) \arrow[u, phantom, "\cong"]
\end{tikzcd}$$

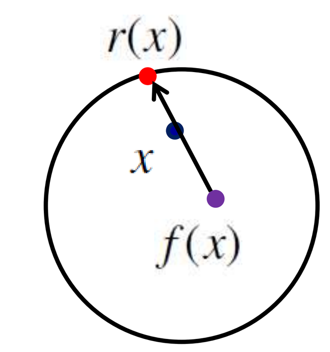
This is a contradiction since this would imply that there exists a surjection from a single point set to all of $\mathbb{Z}$. Hence, there can be no retraction of the circle to the disk. ■

Fixed Point Theorem of Brouwer

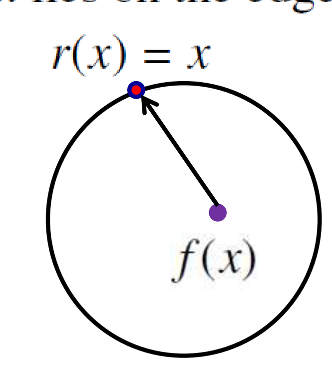
Every continuous function mapping D^2 to itself has a fixed point.

Proof: Let $f : D^2 \to D^2$ be continuous. Proceeding by contradiction, suppose there is no fixed point of f. In other words, for every $x \in D^2$, $f(x) \neq x$.

Define a new function $r : D^2 \to D^2$. For $x \in D^2$, let $r(x)$ be the tip of the ray on S^1 that begins at $f(x)$ and passes through x. Note that r is well-defined since $f(x) \neq x$ means that $r(x) \in S^1$ for all $x \in D^2$. Since f is continuous it follows that r is continuous. $r(x)$ is pictured below:



But what happens when x lies on the edge of the circle?



$r(x) = x$ when $x \in S^1$.

Hence, $r|_{S^1} = id_{S^1}$. So r is actually a retraction of S^1 to D^2. This is a contradiction by the previous lemma since no retraction of the circle to the disk can exist. Therefore, every continuous function mapping the disk to itself has a fixed point. ■

Applications

Economics: The Fixed Point Theorem of Brouwer can be used to show economic equilibrium exists in certain economic models. In other words, given the proper assumptions, there is always a point where quantity demanded is equal to the quantity supplied. (Adams and Franzosa).

Game Theory: Brouwer's Fixed Point Theorem can be used to show the existence of a Nash equilibrium in certain games. Informally, a Nash equilibrium occurs in a game when each player cannot benefit by changing his or her strategy. (Wikipedia, "Nash Equilibrium". 3-13-2012).

Linear Algebra: The Fixed Point Theorem of Brouwer can be used to show that every 3×3 matrix of positive real numbers has a positive real eigenvalue. In addition, this theorem can show the existence of solutions to certain systems of equations. (Munkres).

References

Adams, Colin and Robert Franzosa. *Introduction to Topology: Pure and Applied*. Prentice Hall, 2008.

May, J.P. *A Concise Course in Algebraic Topology*. The University of Chicago Press, 1999.

Munkres, James. *Topology*. Prentice Hall, 2000.