Waldhausen Additivity and Approximation in Quasicategorical K-Theory

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Motivation

1. The Additivity Theorems of Quillen and Waldhausen are fundamental theorems in K-theory, many other results follow.

2. Connective algebraic K-theory can be universally characterized in terms of Additivity and other properties (Blumberg-Gepner-Tabuada, Barwick)

Goal: direct proofs of Additivity and Approximation entirely in the setting of quasicategories, with general classes of cofibrations

Along the way: results on adjunctions between quasicategories.

Additivity and Approximation

Main Result: Additivity

Theorem (Additivity, F.-Lück)

Suppose a split-exact sequence of Waldhausen quasicategories and exact functors

$$\mathcal{A} \underset{j}{\overset{i}{\rightleftarrows}} \mathcal{E} \underset{g}{\overset{f}{\rightleftarrows}} \mathcal{B}$$

satisfies some mild hypotheses. Then the following hold.

1. *The map*

$$S_\bullet^\infty(j,f)_{\mathrm{equiv}} \colon (S_\bullet^\infty \mathcal{E})_{\mathrm{equiv}} \longrightarrow (S_\bullet^\infty \mathcal{A})_{\mathrm{equiv}} \times (S_\bullet^\infty \mathcal{B})_{\mathrm{equiv}}$$

   is a diagonal weak equivalence of bisimplicial sets.

2. *The functors j and f induce a stable equivalence of K-theory spectra*

$$K(j,f) \colon K(\mathcal{E}) \longrightarrow K(\mathcal{A}) \vee K(\mathcal{B}).$$

Main Result: Approximation (General Cofibrations)

Theorem (Approximation 1, F.)

Let G be an exact functor between Waldhausen quasicategories. Suppose:

1. *The functor G induces an equivalence of cofibration homotopy categories.*

Then G induces a stable equivalence of K-theory spectra.

Theorem (Approximation 2, F.)

Let G be an exact functor between Waldhausen quasicategories. Suppose:

1. *The functor G induces an equivalence of homotopy categories.*
2. *The functor G reflects cofibrations.*

Then G induces a stable equivalence of K-theory spectra.

Main Result: Approximation (General Cofibrations)

Theorem (Approximation 1, F.)

Let G be an exact functor between Waldhausen quasicategories. Suppose:

1. *The functor G induces an equivalence of* **cofibration homotopy categories**.

Then G induces a stable equivalence of K-theory spectra.

Theorem (Approximation 2, F.)

Let G be an exact functor between Waldhausen quasicategories. Suppose:

1. *The functor G induces an equivalence of* **homotopy categories**.
2. *The functor G reflects cofibrations.*

Then G induces a stable equivalence of K-theory spectra.

Main Technical Result to Prove Approximation

Proposition (F.)

Let $F : \mathcal{A} \to \mathcal{B}$ be a functor between quasicategories (no Waldhausen structure is assumed). Suppose:

1. *The overquasicategory $F \downarrow b$ is connected for every $b \in B$.*
2. *The functor F is essentially surjective.*
3. *The quasicategory $\mathcal{A}$ admits all colimits of diagrams in $\mathcal{A}_{\mathrm{equiv}}$ indexed by finite (not necessarily connected) posets, and F preserves such colimits. Note that $\mathcal{A}_{\mathrm{equiv}}$ itself is not assumed to admit such colimits.*
4. *F reflects equivalences.*

Then $F_{\mathrm{equiv}} : \mathcal{A}_{\mathrm{equiv}} \to \mathcal{B}_{\mathrm{equiv}}$ is a weak homotopy equivalence.

Waldhausen Quasicategories

Definition (Waldhausen quasicategory, F.-L.)

A *Waldhausen quasicategory* consists of a quasicategory $\mathcal{C}$ together with a distinguished zero object $*$ and a subquasicategory $co\mathcal{C}$, the 1-simplices of which are called *cofibrations* and denoted $\rightarrowtail$, such that

1. The subquasicategory $co\mathcal{C}$ is 1-full in $\mathcal{C}$ and contains all equivalences in $\mathcal{C}$,
2. For each object A of $\mathcal{C}$, every morphism $* \to A$ is a cofibration,
3. The pushout of a cofibration along any morphism exists, and every pushout of a cofibration along any morphism is a cofibration.

Examples of Waldhausen Quasicategories

1. Classical Waldhausen categories produce Waldhausen quasicategories (Barwick).

2. The quasicategory of modules for an A_∞ ring spectrum is a Waldhausen quasicategory.

3. Any *stable* quasicategory is a Waldhausen quasicategory in which all maps are cofibrations.

Remarks on the Notion of Waldhausen Quasicategory

Let $\mathcal{C}$ be a Waldhausen quasicategory.

1. $co\mathcal{C} \supseteq \mathcal{C}_{\text{equiv}}$
2. "weak equivalences" are the equivalences
3. Gluing Lemma automatically holds
4. $\tau_1 \left(co\mathcal{C} \right) \subseteq \tau_1 \mathcal{C}$
5. Our definition is equivalent to the definition of Barwick.

Classical Hypotheses are Resituated for Waldhausen Quasicategories

Let $\mathcal{C}$ be a Waldhausen quasicategory. Then

1. 3-for-2, or "saturation", automatically holds
2. any map homotopic to a cofibration is a cofibration
3. "factorization" implies every map is a cofibration
4. "weak cofibrations" of Blumberg–Mandell are same as cofibrations
5. "homotopy cocartesian squares" of Blumberg–Mandell are the same as pushouts with one leg a cofibration
6. statements are in general easier to formulate.

Sketch of Proof of Additivity

Idea: Reprove classical Waldhausen Additivity in a simplicial way, and transfer the proof to quasicategories.

Reduce to case $\mathcal{A} \underset{s}{\overset{i}{\rightleftarrows}} \mathcal{E}(\mathcal{A},\mathcal{C},\mathcal{B}) \underset{g}{\overset{q}{\rightleftarrows}} \mathcal{B}$ where $\mathcal{E}(\mathcal{A},\mathcal{C},\mathcal{B})$ is the quasicategory of cofiber sequences $A \rightarrowtail C \twoheadrightarrow B$ and $\mathcal{A},\mathcal{B} \subseteq \mathcal{C}$, and $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

Let $\mathfrak{s}_\bullet\mathcal{C} := \mathrm{Obj}\ S_\bullet\mathcal{C}$ and $(f,g) = (\mathfrak{s}_\bullet s, \mathfrak{s}_\bullet q) : \mathfrak{s}_\bullet\mathcal{E} \to \mathfrak{s}_\bullet\mathcal{A} \times \mathfrak{s}_\bullet\mathcal{B}$ in **SSet**. Then

$$f/(m,A') \xrightarrow{\text{proj}} \mathfrak{s}_\bullet\mathcal{E} \xrightarrow{g} \mathfrak{s}_\bullet\mathcal{B}$$

is a weak equivalence of simplicial sets for all $A' \in \mathfrak{s}_m\mathcal{A}$ (work by F.-Lück, McCarthy, Rognes, Waldhausen).

Theorem $\hat{\mathrm{A}}^*$ of Grayson et al. $\Rightarrow$ (f,g) is a weak equivalence of simplicial sets. Repeat for sequences, and use Realization Lemma to get a weak equivalence of bisimplicial sets

$$wS_\bullet(s,q) : wS_\bullet\mathcal{E} \to wS_\bullet\mathcal{A} \times wS_\bullet\mathcal{B}. \quad \square$$

Reduction to $\mathcal{E}(\mathcal{A}, \mathcal{C}, \mathcal{B})$ for Quasicategories??

Adjunctions between Quasicategories and Homotopical Adjunctions of Simplicial Categories

The 2-Category of Simplicial Sets, Joyal

Let $\tau_1 : \mathbf{SSet} \to \mathbf{Cat}$ be the left adjoint to nerve.
If X is a quasicategory, $\tau_1 X$ is isomorphic to the homotopy category of X:
$\tau_1(X)$ has objects X_0,
$\tau_1(X)(x, y)$ is homotopy classes of maps $x \to y$.

The 2-category of simplicial sets is $\mathbf{SSet}^{\tau_1}$:
objects are simplicial sets $A, B, \ldots$
$\mathbf{SSet}^{\tau_1}(A, B)$ is the category $\tau_1(B^A)$.

Adjunctions between Quasicategories, Joyal

Definition (Joyal)

An *adjunction between quasicategories* is an adjunction between quasicategories in the 2-category $\mathbf{SSet}^{\tau_1}$.
This consists of functors $f\colon \mathcal{C} \to \mathcal{D}$ and $g\colon \mathcal{D} \to \mathcal{C}$,
and natural transformations $\eta\colon \mathrm{Id}_{\mathcal{C}} \to g \circ f$ and $\varepsilon\colon f \circ g \to \mathrm{Id}_{\mathcal{D}}$,
such that the triangle identities hold in the 2-category $\mathbf{SSet}^{\tau_1}$:
$(g * [\varepsilon]) \odot ([\eta] * g) = \mathrm{Id}_g \qquad ([\varepsilon] * f) \odot (f * [\eta]) = \mathrm{Id}_f$.

Homotopy Natural Transformations in **SimpCat**

Simplicial categories, simplicial functors, and homotopy natural transformations form a sesquicategory **SimpCat**.

A *homotopy natural transformation* $\alpha\colon f \Rightarrow g$ *of simplicial functors* $\mathcal{C} \to \mathcal{D}$ assigns to each object x a morphism $\alpha_x\colon fx \to gx$ which is *homotopy-natural*, that is, the two composite maps of simplicial sets

$$\mathcal{C}(x,y) \xrightarrow{f_{x,y}} \mathcal{D}(fx,fy) \xrightarrow{(\alpha_y)_*} \mathcal{D}(fx,gy)$$

$$\mathcal{C}(x,y) \xrightarrow{g_{x,y}} \mathcal{D}(gx,gy) \xrightarrow{(\alpha_x)^*} \mathcal{D}(fx,gy)$$

have homotopic geometric realizations.

Homotopy Adjunctions between Simplicial Categories

A *homotopy adjunction between simplicial categories* consists of simplicial functors $f : \mathcal{C} \to \mathcal{D}$ and $g : \mathcal{D} \to \mathcal{C}$
and homotopy natural transformations $\eta : 1_{\mathcal{C}} \Rightarrow gf$ and
$\varepsilon : fg \Rightarrow 1_{\mathcal{D}}$
such that the components of the triangle identities

$$(g * \varepsilon) \odot (\eta * g) = \mathsf{Id}_g \qquad (\varepsilon * f) \odot (f * \eta) = \mathsf{Id}_f$$

are true in $\pi_0\mathcal{C}$ and $\pi_0\mathcal{D}$ respectively.

Proposition

In **SimpCat**, *a homotopy right adjoint is homotopically fully faithful (w.e. on hom ssets) if and only if every component of the counit ε is an equivalence. Consequently, an analogous result holds for* **quasicategories**.

Reduction to $\mathcal{E}(\mathcal{A}, \mathcal{C}, \mathcal{B})$ for Quasicategories

Proposition (Reduction)

Suppose a split-exact sequence of Waldhausen quasicategories and exact functors

$$\mathcal{A} \underset{j}{\overset{i}{\rightleftarrows}} \mathcal{E} \underset{g}{\overset{f}{\rightleftarrows}} \mathcal{B}$$

has the following three properties.

1. *Each counit component $ij(E) \to E$ is a cofibration.*
2. *For each morphism $E \to E'$ in $\mathcal{E}$, the induced map*

   $$E \cup_{ij(E)} ij(E') \to E'$$

   is a cofibration in $\mathcal{E}$.
3. *In every cofiber sequence in $\mathcal{A}$ of the form $A_0 \rightarrowtail A_1 \twoheadrightarrow *$, the first map is an equivalence.*

Then it is Waldhausen equivalent to $\mathcal{E}(\mathcal{A}, \mathcal{C}, \mathcal{B})$.

Main Result: Additivity

Theorem (Additivity, F.-Lück)

Suppose a split-exact sequence of Waldhausen quasicategories and exact functors

$$\mathcal{A} \underset{j}{\overset{i}{\rightleftarrows}} \mathcal{E} \underset{g}{\overset{f}{\rightleftarrows}} \mathcal{B}$$

satisfies the hypotheses on the previous slide. Then:

1. *The map*

$$S_\bullet^\infty(j,f)_{\text{equiv}} \colon (S_\bullet^\infty\mathcal{E})_{\text{equiv}} \longrightarrow (S_\bullet^\infty\mathcal{A})_{\text{equiv}} \times (S_\bullet^\infty\mathcal{B})_{\text{equiv}}$$

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