Waldhausen Additivity and Approximation in Quasicategorical *K*-Theory

Thomas M. Fiore partly joint with Wolfgang Lück,

http://www-personal.umd.umich.edu/~tmfiore/ http://www.him.uni-bonn.de/lueck/

- The Additivity Theorems of Quillen and Waldhausen are fundamental theorems in *K*-theory, many other results follow.
- Connective algebraic K-theory can be universally characterized in terms of Additivity and other properties (Blumberg-Gepner-Tabuada, Barwick)

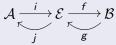
Goal: direct proofs of Additivity and Approximation entirely in the setting of quasicategories, with general classes of cofibrations

Along the way: results on adjunctions between quasicategories.

Additivity and Approximation

Theorem (Additivity, F.-Lück)

Suppose a split-exact sequence of Waldhausen quasicategories and exact functors



satisfies some mild hypotheses. Then the following hold.

The map

$$S^{\infty}_{\bullet}(j,f)_{\text{equiv}} \colon (S^{\infty}_{\bullet}\mathcal{E})_{\text{equiv}} \longrightarrow (S^{\infty}_{\bullet}\mathcal{A})_{\text{equiv}} \times (S^{\infty}_{\bullet}\mathcal{B})_{\text{equiv}}$$

is a diagonal weak equivalence of bisimplicial sets.

2 The functors j and f induce a stable equivalence of K-theory spectra

$$K(j, f) \colon K(\mathcal{E}) \longrightarrow K(\mathcal{A}) \vee K(\mathcal{B}).$$

Main Result: Approximation (General Cofibrations)

Theorem (Approximation 1, F.)

Let G be an exact functor between Waldhausen quasicategories. Suppose:

• The functor G induces an equivalence of cofibration homotopy categories.

Then G induces a stable equivalence of K-theory spectra.

Theorem (Approximation 2, F.)

Let G be an exact functor between Waldhausen quasicategories. Suppose:

- The functor G induces an equivalence of homotopy categories.
- **2** The functor G reflects cofibrations.

Then G induces a stable equivalence of K-theory spectra.

Main Result: Approximation (General Cofibrations)

Theorem (Approximation 1, F.)

Let G be an exact functor between Waldhausen quasicategories. Suppose:

• The functor G induces an equivalence of cofibration homotopy categories.

Then G induces a stable equivalence of K-theory spectra.

Theorem (Approximation 2, F.)

Let G be an exact functor between Waldhausen quasicategories. Suppose:

- The functor G induces an equivalence of homotopy categories.
- **2** The functor *G* reflects cofibrations.

Then G induces a stable equivalence of K-theory spectra.

Proposition (F.)

Let $F : A \rightarrow B$ be a functor between quasicategories (no Waldhausen structure is assumed). Suppose:

- The overquasicategory $F \downarrow b$ is connected for every $b \in B$.
- **2** The functor *F* is essentially surjective.
- The quasicategory A admits all colimits of diagrams in A_{equiv} indexed by finite (not necessarily connected) posets, and F preserves such colimits. Note that A_{equiv} itself is not assumed to admit such colimits.
- **④** *F* reflects equivalences.

Then $F_{\rm equiv} : \mathcal{A}_{\rm equiv} \to \mathcal{B}_{\rm equiv}$ is a weak homotopy equivalence.

Definition (Waldhausen quasicategory, F.-L.)

A Waldhausen quasicategory consists of a quasicategory C together with a distinguished zero object * and a subquasicategory coC, the 1-simplices of which are called *cofibrations* and denoted \rightarrow , such that

- The subquasicategory *coC* is 1-full in *C* and contains all equivalences in *C*,
- **2** For each object A of C, every morphism $* \to A$ is a cofibration,
- The pushout of a cofibration along any morphism exists, and every pushout of a cofibration along any morphism is a cofibration.

- Classical Waldhausen categories produce Waldhausen quasicategories (Barwick).
- 2 The quasicategory of modules for an A_{∞} ring spectrum is a Waldhausen quasicategory.
- Any stable quasicategory is a Waldhausen quasicategory in which all maps are cofibrations.

Remarks on the Notion of Waldhausen Quasicategory

Let \mathcal{C} be a Waldhausen quasicategory.

- $coC \supseteq C_{equiv}$
- Weak equivalences are the equivalences
- Gluing Lemma automatically holds
- Our definition is equivalent to the definition of Barwick.

Classical Hypotheses are Resituated for Waldhausen Quasicategories

Let $\ensuremath{\mathcal{C}}$ be a Waldhausen quasicategory. Then

- 3-for-2, or "saturation", automatically holds
- 2 any map homotopic to a cofibration is a cofibration
- **③** "factorization" implies every map is a cofibration
- "weak cofibrations" of Blumberg–Mandell are same as cofibrations
- (a) "homotopy cocartesian squares" of Blumberg–Mandell are the same as pushouts with one leg a cofibration
- **o** statements are in general easier to formulate.

Sketch of Proof of Additivity

Idea: Reprove classical Waldhausen Additivity in a simplicial way, and transfer the proof to quasicategories.

Reduce to case
$$\mathcal{A} \xrightarrow{i} \mathcal{E}(\mathcal{A}, \mathcal{C}, \mathcal{B}) \xrightarrow{q} \mathcal{B}$$
 where $\mathcal{E}(\mathcal{A}, \mathcal{C}, \mathcal{B})$ is

the quasicategory of cofiber sequences $A \rightarrow C \twoheadrightarrow B$ and $\mathcal{A}, \mathcal{B} \subseteq \mathcal{C}$, and $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

Let $\mathfrak{s}_{\bullet}\mathcal{C} := \text{Obj } S_{\bullet}\mathcal{C}$ and $(f,g) = (\mathfrak{s}_{\bullet}s, \mathfrak{s}_{\bullet}q) : \mathfrak{s}_{\bullet}\mathcal{E} \to \mathfrak{s}_{\bullet}\mathcal{A} \times \mathfrak{s}_{\bullet}\mathcal{B}$ in SSet. Then

$$f/(m, A') \xrightarrow{\operatorname{proj}} \mathfrak{s}_{\bullet} \mathcal{E} \xrightarrow{g} \mathfrak{s}_{\bullet} \mathcal{B}$$

is a weak equivalence of simplicial sets for all $A' \in \mathfrak{s}_m \mathcal{A}$ (work by F.-Lück, McCarthy, Rognes, Waldhausen).

Theorem \hat{A}^* of Grayson et al. \Rightarrow (f,g) is a weak equivalence of simplicial sets. Repeat for sequences, and use Realization Lemma to get a weak equivalence of bisimplicial sets

$$wS_{\bullet}(s,q): wS_{\bullet}\mathcal{E} \to wS_{\bullet}\mathcal{A} \times wS_{\bullet}\mathcal{B}.$$

Reduction to $\mathcal{E}(\mathcal{A}, \mathcal{C}, \mathcal{B})$ for Quasicategories??

Adjunctions between Quasicategories and Homotopical Adjunctions of Simplicial Categories

```
Let \tau_1 : \mathbf{SSet} \to \mathbf{Cat} be the left adjoint to nerve.
If X is a quasicategory, \tau_1 X is isomorphic to the homotopy category of X:
\tau_1(X) has objects X_0,
\tau_1(X)(x, y) is homotopy classes of maps x \to y.
```

```
The 2-category of simplicial sets is SSet<sup>\tau_1</sup>: objects are simplicial sets A, B, ...
SSet<sup>\tau_1</sup>(A, B) is the category \tau_1(B^A).
```

Definition (Joyal)

An *adjunction between quasicategories* is an adjunction between quasicategories in the 2-category \mathbf{SSet}^{τ_1} . This consists of functors $f: \mathcal{C} \to \mathcal{D}$ and $g: \mathcal{D} \to \mathcal{C}$, and natural transformations $\eta: \mathrm{ld}_{\mathcal{C}} \to g \circ f$ and $\varepsilon: f \circ g \to \mathrm{Id}_{\mathcal{D}}$, such that the triangle identities hold in the 2-category \mathbf{SSet}^{τ_1} : $(g * [\varepsilon]) \odot ([\eta] * g) = \mathrm{Id}_g \quad ([\varepsilon] * f) \odot (f * [\eta]) = \mathrm{Id}_f$. Simplicial categories, simplicial functors, and homotopy natural transformations form a sesquicategory **SimpCat**.

A homotopy natural transformation $\alpha: f \Rightarrow g$ of simplicial functors $\mathcal{C} \to \mathcal{D}$ assigns to each object x a morphism $\alpha_x: fx \to gx$ which is homotopy-natural, that is, the two composite maps of simplicial sets

$$\mathcal{C}(x,y) \xrightarrow{f_{x,y}} \mathcal{D}(fx,fy) \xrightarrow{(\alpha_y)_*} \mathcal{D}(fx,gy)$$

$$\mathcal{C}(x,y) \xrightarrow{g_{x,y}} \mathcal{D}(gx,gy) \xrightarrow{(\alpha_x)^*} \mathcal{D}(fx,gy)$$

have homotopic geometric realizations.

Homotopy Adjunctions between Simplicial Categories

A homotopy adjunction between simplicial categories consists of simplicial functors $f: \mathcal{C} \to \mathcal{D}$ and $g: \mathcal{D} \to \mathcal{C}$ and homotopy natural transformations $\eta: 1_{\mathcal{C}} \Rightarrow gf$ and $\varepsilon: fg \Rightarrow 1_{\mathcal{D}}$

such that the components of the triangle identities

$$(g * \varepsilon) \odot (\eta * g) = \mathsf{Id}_g \quad (\varepsilon * f) \odot (f * \eta) = \mathsf{Id}_f$$

are true in $\pi_0 C$ and $\pi_0 D$ respectively.

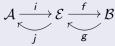
Proposition

In **SimpCat**, a homotopy right adjoint is homotopically fully faithful (w.e. on hom ssets) if and only if every component of the counit ε is an equivalence. Consequently, an analogous result holds for **quasicategories**.

Reduction to $\mathcal{E}(\mathcal{A}, \mathcal{C}, \mathcal{B})$ for Quasicategories

Proposition (Reduction)

Suppose a split-exact sequence of Waldhausen quasicategories and exact functors



has the following three properties.

- Each counit component $ij(E) \rightarrow E$ is a cofibration.
- 2 For each morphism $E \to E'$ in \mathcal{E} , the induced map

 $E \cup_{ij(E)} ij(E') \rightarrow E'$

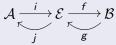
is a cofibration in \mathcal{E} .

③ In every cofiber sequence in A of the form $A_0 → A_1 → *$, the first map is an equivalence.

Then it is Waldhausen equivalent to $\mathcal{E}(\mathcal{A}, \mathcal{C}, \mathcal{B})$.

Theorem (Additivity, F.-Lück)

Suppose a split-exact sequence of Waldhausen quasicategories and exact functors



satisfies the hypotheses on the previous slide. Then:

The map

$$S^{\infty}_{\bullet}(j,f)_{\text{equiv}} \colon (S^{\infty}_{\bullet}\mathcal{E})_{\text{equiv}} \longrightarrow (S^{\infty}_{\bullet}\mathcal{A})_{\text{equiv}} \times (S^{\infty}_{\bullet}\mathcal{B})_{\text{equiv}}$$

is a diagonal weak equivalence of bisimplicial sets.

The functors j and f induce a stable equivalence of K-theory spectra

$$K(j, f) \colon K(\mathcal{E}) \longrightarrow K(\mathcal{A}) \vee K(\mathcal{B}).$$