Mathematics and Music
2007 REU
Topos of Triads Problems
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1. Recall that a monoid is a set with an associative and unital binary operation, and an affine map $\mathbb{Z}_{12} \longrightarrow \mathbb{Z}_{12}$ is a map of the form ${}^t s(z) = sz + t$. Prove that the set of affine maps $\mathbb{Z}_{12} \longrightarrow \mathbb{Z}_{12}$ is a monoid.

2. Prove that an affine map $\mathbb{Z}_{12} \longrightarrow \mathbb{Z}_{12}$ only depends on its values at 0 and 1. Given an affine map, determine s and t from the values at 0 and 1. How many elements does the monoid of affine maps $\mathbb{Z}_{12} \longrightarrow \mathbb{Z}_{12}$ have?

3. (a) Prove that a monoid is the same thing as a one-object category.
(b) Prove that a set with an action by a monoid M is the same as a functor $M \longrightarrow Sets$. What do left and right actions correspond to?
(c) Prove that an M-equivariant map is the same as a natural transformation between the associated functors.
(d) Explain why we use the notation $Sets^{\mathcal{T}}$ for the topos of triads.

4. Prove that the inclusion $\{0\} \longrightarrow \{0, 1\} = \Omega$ is a subobject classifier in the category of sets and functions. The characteristic morphisms are the characteristic functions, i.e., $\chi_{A \subseteq X}(x) = 0$ if and only if $x \in A$ and is 1 otherwise. Compare this problem with the topos axioms.

5. In the category of sets and functions, prove that there is a natural bijection

$$Hom(X \times Y, Z) \cong Hom(X, Map(Y, Z))$$

$$f \mapsto (x \mapsto f(x, -)).$$

In the category of sets and functions, $Map(Y, Z)$ is the same as $Hom(Y, Z)$. Compare this problem with the topos axioms.

6. Recall that the triadic monoid is $\mathcal{T} = \{{}^0 0, {}^1 0, {}^4 0, {}^0 1, {}^1 3, {}^4 9, {}^4 8, {}^0 4\}$ where ${}^t s$ denotes the affine map $\mathbb{Z}_{12} \longrightarrow \mathbb{Z}_{12}$ given by ${}^t s(z) = sz + t$. Prove that this monoid is precisely the the monoid of affine maps preserving the C-major triad $\{0, 1, 4\}$. (We are using circle of fifths encoding). Hint: Recall that ${}^t s$ is determined by what it does to 0 and 1.

7. Recall that if A is a subobject of X in the topos of triads $Sets^{\mathcal{T}}$, its characteristic morphism $\chi_A : X \longrightarrow \Omega$ is defined by

$$\chi_A(x) := \{h \in \mathcal{T} | hx \in A\}$$

where $\Omega = \{\mathcal{T}, \mathcal{P}, \mathcal{L}, \mathcal{R}, \mathcal{C}, \mathcal{F}\}$. Calculate the characteristic function $j_{\mathcal{P}}$ for the subobject $\{\mathcal{P}, \mathcal{T}\}$ of Ω. Hint: Use the multiplication table on page 112 and the lattice on page 114 of Noll's article. To check your answer, here is a table.

x	$j_{\mathcal{P}}(x)$	$j_{\mathcal{L}}(x)$	$j_{\mathcal{R}}(x)$
$\mathcal{F}$	$\mathcal{F}$	$\mathcal{F}$	$\mathcal{F}$
$\mathcal{C}$	$\mathcal{C}$	$\mathcal{R}$	$\mathcal{L}$
$\mathcal{L}$	$\mathcal{L}$	$\mathcal{T}$	$\mathcal{L}$
$\mathcal{R}$	$\mathcal{R}$	$\mathcal{R}$	$\mathcal{T}$
$\mathcal{P}$	$\mathcal{T}$	$\mathcal{T}$	$\mathcal{T}$
$\mathcal{T}$	$\mathcal{T}$	$\mathcal{T}$	$\mathcal{T}$

The subobject classifier in the topos of triads $Sets^{\mathcal{T}}$ is the inclusion $\{\mathcal{T}\} \longrightarrow \Omega$.

8. Find the subobject of $\mu[10,4] : \mathcal{T} \times \mathbb{Z}_{12} \longrightarrow \mathbb{Z}_{12}$ whose characteristic morphism is $j_{\mathcal{L}} \circ \chi_{\{0,10,4\}}$. Identify the type of scale. Hint: Use the tables on the first two slides to calculate the preimage of $\mathcal{T}$.