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The topos of triads. (English summary)
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Thomas Noll presents in this article several fascinating music-theoretical interpretations of the topos $\mathrm{Sets}^{\mathcal{T}}$, as well as an analysis of Scriabin's Étude Number 3, Opus 65.

The notion of elementary topos simultaneously generalizes the notions of topological space and universe of sets, reflecting its separate origins: Grothendieck's work in algebraic geometry and Lawvere's foundational investigations, both from the early 1960's [see S. Mac Lane and I. Moerdijk, in *Handbook of algebra, Vol. 1*, 501–528, North-Holland, Amsterdam, 1996; MR1421810 (97j:18005)]. This common generalization has been extremely useful in the mathematical realms of algebraic geometry and set theory. Recently, G. Mazzola and collaborators have developed a topos-theoretic approach to music theory [G. Mazzola, *The topos of music*. In collaboration with Stefan Göller and Stefan Müller. Birkhäuser, Basel, 2002; MR1938949 (2004a:00013)], and this has inspired Noll's work.

An elementary topos, or simply topos, is a category with finite limits and a subobject classifier such that the functor $B \times -$ for each object B admits a right adjoint. Examples include the category of sheaves on a topological space X and the category Sets of sets and functions. A subobject classifier is an object Ω equipped with a morphism from the terminal object to Ω which is a universal monomorphism: any other monomorphism is a pullback in a unique way. Thus, for any object B there is a bijection between subobjects $A \to B$ and morphisms $B \to \Omega$. The morphism corresponding to a subobject is called its characteristic morphism. In the case of Sets, a subobject is a subset, the subobject classifier is the inclusion $\{1\} \to \{0,1\}$, and characteristic morphisms are the usual characteristic functions.

The present article is concerned with the topos of triads $\mathrm{Sets}^{\mathcal{T}}$. If we encode the twelve pitch classes as Z_{12} via the circle of fifths, i.e. $0 = C$, $1 = G$, $2 = D, \dots,$ then the C-major triad $\{C, E, G\}$ is $\{0, 1, 4\}$. There are eight affine transformations $z \mapsto sz + t$ that preserve the set $\{0, 1, 4\}$, and these form the monoid $\mathcal{T}$. We denote by $\mathrm{Sets}^{\mathcal{T}}$ the category of sets equipped with a left $\mathcal{T}$-action and $\mathcal{T}$-equivariant functions. For each $m, n \in Z_{12}$ we have an object $\mu[m, n]$: this is the action of $\mathcal{T}$ on Z_{12} where the generators f and g act by $z \mapsto 3z + m$ and $z \mapsto 8z + n$ respectively. In particular, $\mu[1, 4]$ is the natural action of $\mathcal{T}$ on Z_{12}. Noll explicitly determines the subobject classifier Ω of $\mathrm{Sets}^{\mathcal{T}}$, the subobjects of Ω, and their characteristic morphisms. As is well known, Ω is the set of left ideals in $\mathcal{T}$, three of which Noll judiciously calls $\mathcal{P}$, $\mathcal{L}$, and $\mathcal{R}$.

As is not well known, this actually has music-theoretical relevance. Given a left ideal $\mathcal{B} \in \Omega$, the upgrade $j_{\mathcal{B}}$ of $\mathcal{B}$ is the characteristic morphism $\Omega \to \Omega$ associated to the subobject $\{\mathcal{B}' | \, \mathcal{B} \subseteq \mathcal{B}'\}$ of Ω. Each of these morphisms satisfies three axioms making them into Lawvere-Tierney topologies. If we denote by $\chi_{\{0,1,4\}}$ the characteristic morphism for the C-major triad as a subobject of $\mu[1, 4]$, then the subobjects corresponding to $j_{\mathcal{P}} \circ \chi_{\{0,1,4\}}$, $j_{\mathcal{L}} \circ \chi_{\{0,1,4\}}$, and $j_{\mathcal{R}} \circ \chi_{\{0,1,4\}}$ are respectively $\{C, E, E\flat, G\}$, a hexatonic cycle, and an octatonic scale. Noll refers to these sets as upgraded triads of $\{0,1,4\}$. Since Lawvere-Tierney topologies can be regarded as modal operators, Noll views the upgraded triads as "modalities where the original triad is still locally present". They consist of course precisely of those elements of Z_{12} such that $j \circ \chi_{\{0,1,4\}}$ takes the value $\mathcal{T}$ ("true").

One can similarly upgrade the stretched triad $\{0,4,10\}$ as a subobject of $\mu[10,4]$. In this case, one obtains the French augmented sixth chord, the even whole tone scale, and an octatonic scale respectively. The upgraded triads of $\{0+k, 4+k, 10+k\}$ as a subobject of $\mu[10-2k, 4-7k]$ are the transpositions of the upgraded triads for $\{0,4,10\}$ by k, since the characteristic morphism of $\{0+k, 4+k, 10+k\}$ is an appropriate transposition of the characteristic morphism for $\{0,4,10\}$.

Nearly all of the left-hand chords of Scriabin's Étude Number 3, Opus 65 are of the form $\{0+k, 4+k, 10+k\}$, and Noll partitions the Étude into harmonic cells accordingly. The added right-hand tones in each cell are regarded as "local dissonances", and Noll collates the values of these local dissonances and the bass note under the respective characteristic morphisms $Z_{12} \to \Omega$ into a table. He also provides a diagram illustrating membership of the bass note in the various upgraded triads of the large scale opening stretched triad $\{3,7,1\}$.

From this data, Noll draws several conclusions. The two tones with value equal to the smallest nontrivial left ideal ("lowest truth value") are missing entirely from the piece. The value of the ubiquitous tritone oscillation is the largest proper left ideal (largest "non-true" truth value). The global behavior of the bass with respect to $\{3,7,1\}$ mimics the local behavior of the "local dissonances" in the sense that they all avoid the lowest truth value. Further, the stretched triad $\{3,7,1\}$ is always "locally present" since the bass is always in one of its upgraded triads.

Noll closes the article with a calculation of the minimal j-dense subobjects of the actions $\mu[m,n]$. However, this section raises several questions. Why is it that $E\flat, B, A$ (precisely the notes created by the neo-Riemannian operations P, L, R in minimal voice leading from the C-major triad) are in the complement of the minimal $j_{\mathcal{P}}, j_{\mathcal{L}}, j_{\mathcal{R}}$-dense subobjects of $\mu[1,4]$? Noll outlines some future work in this direction.

Though some readers might consider the article overly succinct, this reviewer highly recommends it. Students of topos theory will enjoy reading an interesting application, but more importantly, the article opens some new avenues in mathematical music theory. Noll makes great use of monoid actions (as opposed to simply transitive group actions), and this provides the key to a mathematical description of Carl Stumpf's conception of the triad.

{For the entire collection see MR2238829 (2007a:00007)}

Reviewed by *Thomas M. Fiore*