Voicing Transformations and a Linear Representation of Uniform Triadic Transformations

Thomas M. Fiore and Thomas Noll

http://www-personal.umd.umich.edu/~tmfiore/

Preprint on arXiv

Preview

Math Results: Structure Theorem for a certain matrix group $\mathcal{J}$ generated by voicing reflections,
the center of $\mathcal{J}$,
the centralizers of $\mathcal{J}$ in $GL(3, \mathbb{Z}_{12})$ and $\text{Aff}(3, \mathbb{Z}_{12})$,
compatibilities of $\mathcal{J}$ with permutations

Motivation: RICH in Joe Straus' interpretation of Webern, Concerto for Nine Instruments, Op. 24, Second Movement

Applications:

- Find a matrix with orbit the diatonic falling fifth sequence
- Find four matrices that realize the *PLP* and *L* flip-flop in Clampitt's interpretation of Grail sequence
- A linear representation of Hook's group of uniform triadic transformations
- More in the paper.

Motivation for Studying the Group $\mathcal{J}$

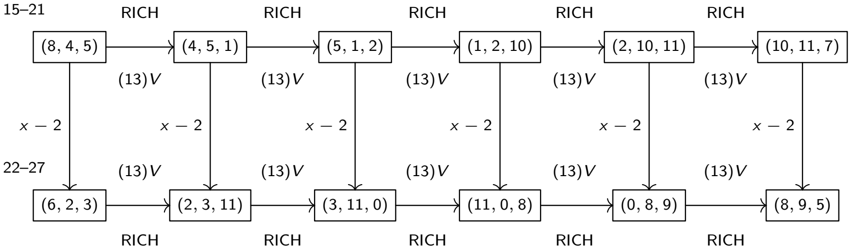
Webern, Concerto for Nine Instruments, Op. 24, Second Movement

Motivation for Studying the Group $\mathcal{J}$
Webern, Concerto for Nine Instruments, Op. 24, Second Movement

The series in this twelve-tone work is constructed from set class 014.

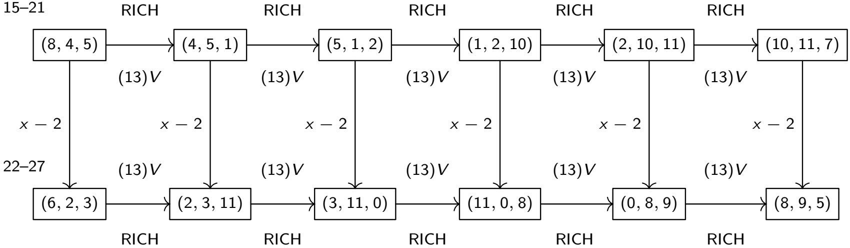
See page 58 of Straus' article *Contextual-Inversion Spaces* for a bracketing of the first 28 measures of the melody into enchained trichords from set class 014.

We draw lines 3 and 4 *in temporal order* as

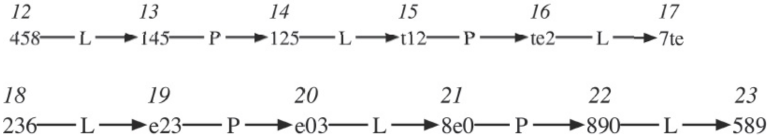


Motivation for Studying the Group $\mathcal{J}$
Webern, Concerto for Nine Instruments, Op. 24, Second Movement

We write:



Straus writes pcsegs non-temporally in normal form and uses “L” and “P” analogues:



Motivation for Studying the Group $\mathcal{J}$
Webern, Concerto for Nine Instruments, Op. 24, Second Movement

Problems with writing "L" and "P" for non-consonant 014 set class:

- we would expect a hexatonic PL-cycle, but this is octatonic, not hexatonic
- there is not an obvious root in pitch-class segments
- the definition of "L" and "P" is complex to define for re-orderings of pitch-class segments (first go to prime form, then operate, then come back)

Advantages of writing $(13)V$ for RICH

- Avoid the above.
- Temporal ordering is preserved, $(13)V$ is exactly the retrograde inversion enchaining.
- Economy of description.

Suggests: Contrast Three Different Kinds of Reflection of Consonant Triads I_n, L, and V

Global Reflection I_n: has globally chosen axis of reflection
$I_n(0,4,7) = -(0,4,7) + (n,n,n)$ is affine

Contextual Reflection L: axis of reflection depends on input chord, look inside triad, find p and q that differ by a minor third, apply I_{p+q}

$L(0,4,7) = I_{4+7}(0,4,7) = (11,7,4)$ $\qquad L(C) = e$
$L(7,4,0) = I_{4+7}(7,4,0) = (4,7,11)$ $\qquad L(C) = e$
reordered

$$L(\sigma(0,4,7)) = \sigma L \sigma^{-1}(\sigma(0,4,7))$$

Note: input voicing has no effect on output's underlying pcset
this description is not in general unambiguous for other pcsegs

Suggests: Contrast Three Different Kinds of Reflection of Consonant Triads I_n, L, and V

Contextual Reflection L**:** axis of reflection depends on input chord, look inside triad, find p and q that differ by a minor third, apply I_{p+q}

$L(0,4,7) = I_{4+7}(0,4,7) = (11,7,4)$ $\quad L(C) = e$

$L(7,4,0) = I_{4+7}(7,4,0) = (4,7,11)$ $\quad L(C) = e$

reordered

$$L(\sigma(0,4,7)) = \sigma L \sigma^{-1}(\sigma(0,4,7))$$

Note: input voicing has no effect on output's underlying pcset

this description is not in general unambiguous for other pcsegs

Voicing Reflection V**:** axis of reflection depends on tones in 2nd and 3rd voice

$V(0,4,7) = I_{4+7}(0,4,7) = (11,7,4)$ $\quad V(C) = e$

$V(7,4,0) = I_{4+0}(7,4,0) = (9,0,4)$ $\quad V(C) = a$!

Note: input voicing has big effect on output's underlying pcset!

this description is defined on any 3-tuple, and is linear

The Groups $\mathcal{J}$ and $\Sigma_3 \ltimes \mathcal{J}$

The Group $\mathcal{J}$ is Generated by the Voicing Transformations U, V, and W

The voicing transformations

$$U, V, W : \mathbb{Z}_{12}^{\times 3} \longrightarrow \mathbb{Z}_{12}^{\times 3}$$

locally reflect along the axis determined by the tones in two pre-selected voices.

$$\begin{array}{rrcl}
12 & U(x,y,z) := I_{x+y}(x,y,z) & = & (y, x, -z+x+y) \\
23 & V(x,y,z) := I_{y+z}(x,y,z) & = & (-x+y+z, z, y) \\
13 & W(x,y,z) := I_{z+x}(x,y,z) & = & (z, -y+x+z, x)
\end{array}$$

$U, V, W \in GL(3, \mathbb{Z}_{12})$

$$U = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & -1 \end{pmatrix}, \quad V = \begin{pmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad W = \begin{pmatrix} 0 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Definition. The group $\mathcal{J}$ is the subgroup of $GL(3, \mathbb{Z}_{12})$ generated by U, V, and W.

The Group $\Sigma_3 \ltimes \mathcal{J}$ is Generated by Σ_3 and $\mathcal{J}$

Proposition

$\langle \Sigma_3, \mathcal{J} \rangle$ *in* $GL(3, \mathbb{Z}_{12})$ *is* $\Sigma_3 \ltimes \mathcal{J}$.

Example (Interesting Element in $\Sigma_3 \ltimes \mathcal{J}$)

RICH $= (13)V \in \Sigma_3 \ltimes \mathcal{J}$ for all set classes

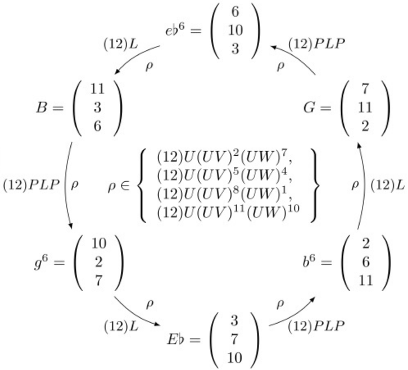
Example (Clampitt's Hexatonic Reading of First Four Chords of Parsifal Grail Motive)

Want in $\Sigma_3 \ltimes \mathcal{J}$:

$$E\flat \xmapsto{PLP} b \xmapsto{L} G \xmapsto{PLP} e\flat$$

Next

Clampitt's Hexatonic Reading of First Four Chords of Parsifal Grail Motive in Orbits of elements in $\Sigma_3 \ltimes \mathcal{J}$



Clampitt's harmonization of Parsifal Grail Motive and a hexatonic extrapolation of Clampitt's hexatonic reading of the first 4 chords.

The hexagon shows the hexatonic extrapolation as an orbit of a single element of $\Sigma_3 \ltimes \mathcal{J}$ in four different ways.
NB: *Exact voicing is produced by all four ρ's!*

The Structure of $\mathcal{J}$

The Structure of $\mathcal{J}$

Theorem (Structure of the Group $\mathcal{J}$, by F.-Noll)

- *The generators U, V, and W satisfy the following relations.*
  1. *Each of U, V, and W has order 2.*
  2. *Both composites UV and UW have order 12.*
  3. *The composite UVW has order 2.*
  4. *The composites UV and UW commute.*
  5. *The U-conjugation of $(UV)^m$ and $(UW)^n$ is inversion.*

$$U^{-1}(UV)^m U = (UV)^{-m} \qquad U^{-1}(UW)^n U = (UW)^{-n}$$

(continues on next slide)

The Structure of $\mathcal{J}$ Continued

Theorem (Structure of the Group $\mathcal{J}$ Continued, F.-Noll)

- *Every element of $\mathcal{J}$ can be written uniquely in the form*

$$U^k(UV)^m(UW)^n \tag{1}$$

where $k = 0, 1$ and $m, n = 0, 1, \ldots, 11$.

- $|\mathcal{J}| = 288$.
- $\mathcal{J} = \langle U \rangle \ltimes \langle UV, UW \rangle \cong \mathbb{Z}_2 \ltimes (\mathbb{Z}_{12} \times \mathbb{Z}_{12})$
where $\mathbb{Z}_2$ acts on $\mathbb{Z}_{12} \times \mathbb{Z}_{12}$ via additive inversion.
- *The elements of $\mathcal{J}$ in the normal form of (1) act as follows*

$$(UV)^m(UW)^n(x,\ y,\ z) = (x,\ y,\ z) + m(z-x) + n(z-y)$$

$$U(UV)^m(UW)^n(x,\ y,\ z) = U(x,\ y,\ z) + m(z-x) + n(z-y)$$

where $\mathbf{x} + c := \mathbf{x} + (c, c, c)$.

First Application of the Structure Theorem: Diatonic Falling Fifth Sequence as an Orbit

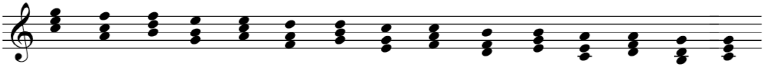
Find a Single Transformation with Orbit the Diatonic Falling Fifth Sequence

To work in C-major, consider $\mathbb{Z}_7$ instead of $\mathbb{Z}_{12}$, and consider $\mathcal{J}(\mathbb{Z}_7)$ instead of $\mathcal{J}$.

Encoding: $C \leftrightarrow 0$, $D \leftrightarrow 1$, $E \leftrightarrow 2$, ..., and finally $B \leftrightarrow 6$

The analogous Structure Theorem for $\mathcal{J}(\mathbb{Z}_7)$ holds.

Question: Use the Structure Theorem to find one matrix in $\Sigma_3 \ltimes \mathcal{J}(\mathbb{Z}_7)$ with orbit the diatonic falling fifth sequence.



$$\begin{pmatrix}0\\2\\4\end{pmatrix} \begin{pmatrix}5\\0\\3\end{pmatrix} \begin{pmatrix}6\\1\\3\end{pmatrix} \begin{pmatrix}4\\6\\2\end{pmatrix} \begin{pmatrix}5\\0\\2\end{pmatrix} \begin{pmatrix}3\\5\\1\end{pmatrix} \begin{pmatrix}4\\6\\1\end{pmatrix} \begin{pmatrix}2\\4\\0\end{pmatrix} \begin{pmatrix}3\\5\\0\end{pmatrix} \begin{pmatrix}1\\3\\6\end{pmatrix} \begin{pmatrix}2\\4\\6\end{pmatrix} \begin{pmatrix}0\\2\\5\end{pmatrix} \begin{pmatrix}1\\3\\5\end{pmatrix} \begin{pmatrix}6\\1\\4\end{pmatrix} \begin{pmatrix}0\\2\\4\end{pmatrix}$$

Find a Single Transformation with Orbit the Diatonic Falling Fifth Sequence

$$\begin{pmatrix}0\\2\\4\end{pmatrix}\begin{pmatrix}5\\0\\3\end{pmatrix}\begin{pmatrix}6\\1\\3\end{pmatrix}\begin{pmatrix}4\\6\\2\end{pmatrix}\begin{pmatrix}5\\0\\2\end{pmatrix}\begin{pmatrix}3\\5\\1\end{pmatrix}\begin{pmatrix}4\\6\\1\end{pmatrix}\begin{pmatrix}2\\4\\0\end{pmatrix}\begin{pmatrix}3\\5\\0\end{pmatrix}\begin{pmatrix}1\\3\\6\end{pmatrix}\begin{pmatrix}2\\4\\6\end{pmatrix}\begin{pmatrix}0\\2\\5\end{pmatrix}\begin{pmatrix}1\\3\\5\end{pmatrix}\begin{pmatrix}6\\1\\4\end{pmatrix}\begin{pmatrix}0\\2\\4\end{pmatrix}$$

Structure Theorem $\Rightarrow$ solve for k, m, n using sequence in

$$U^k(UV)^m(UW)^n(\mathbf{x}) = \mathbf{y}$$

$\Rightarrow$ 2 equations in 2 unknowns

$U(0,2,4) + m(4-0) + n(4-2) = (0,5,3)$

$U(5,0,3) + m(3-5) + n(3-0) = (1,6,3)$

$\Rightarrow$ $m = 3$ and $n = 0$

$\Rightarrow$ $(12)U(UV)^3 = \begin{pmatrix} 5 & 0 & 3 \\ 4 & 1 & 3 \\ 5 & 1 & 2 \end{pmatrix}$ produces the diatonic falling fifth sequence.

Parsifal is similar

Second Application of the Structure Theorem:

A Linear Representation of Hook's Group of Uniform Triadic Transformations

A Linear Representation of Uniform Triadic Transformations

The *uniform triadic transformations* group

$$\mathcal{U} := \{+,-\} \ltimes (\mathbb{Z}_{12} \times \mathbb{Z}_{12})$$

of acts on the set of **abstract** *consonant triads*

$$\mathbb{Z}_{12} \times \{+,-\}.$$

Definition. The *Hook group* $\mathcal{H}$ is the subgroup of $\Sigma_3 \ltimes \mathcal{J}$ that maps root position consonant triads to root position consonant triads.

Proposition

The Hook group $\mathcal{H}$ is generated by the two elements $(1\ 3)U$ and $(1\ 3)W$.

A Linear Representation of Uniform Triadic Transformations

Theorem (Representation of $\mathcal{U}$, by F.-Noll)

The map $\rho\colon \mathcal{U} \to GL(3, \mathbb{Z}_{12})$ *defined by*

$$\rho\langle +, 1, 0\rangle = (UV)^4 (UW)^{-1}$$

$$\rho\langle +, 0, 1\rangle = (UV)^3 (UW)^1$$

$$\rho\langle -, 0, 0\rangle = (1\ 3)W$$

is a representation of $\mathcal{U}$.

It operates on root position triads exactly as uniform triadic transformations would.

The representation ρ *is an isomorphism onto* $\mathcal{H}$.

Summary of Talk

- Structure Theorem for a matrix group $\mathcal{J}$ generated by voicing reflections
- Motivated by expressing RICH as $(13)V$ in Webern, Concerto for Nine Instruments, Op. 24, Second Movement
  - $(13)V = \text{ RICH }$ exactly
  - Economy of description
- Structure Theorem allowed us to find:
  - a matrix with orbit the diatonic falling fifth sequence
  - four matrices that realize the PLP and L flip-flop in Clampitt's interpretation of Grail sequence
  - a linear representation of Hook's group of uniform triadic transformations