Morphisms in a Musical Analysis of Schoenberg, String Quartet No. 1, Opus 7

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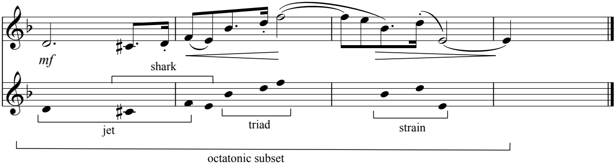
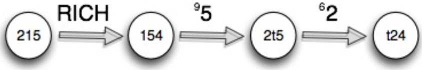
Transformational Analysis of Schoenberg, String Quartet No. 1, Opus 7

Schoenberg, String Quartet No. 1, Opus 7
Opening Theme Materials, Measures 1-3



Octatonic set $\{1, 2, 4, 5, 7, 8, 10, 11\}$ with subsets

Jet $\{2, 1, 5\}$ Shark $\{1, 5, 4\}$	Major $\{10, 2, 5\}$	Strain $\{10, 2, 4\}$



$^{9}5 : \mathbb{Z}_{12} \to \mathbb{Z}_{12}$ denotes the affine function $x \mapsto 5x + 9$

Schoenberg, String Quartet No. 1, Opus 7 End of First Section, Measures 88 - 92, Major/Minor *RP*-Cycle

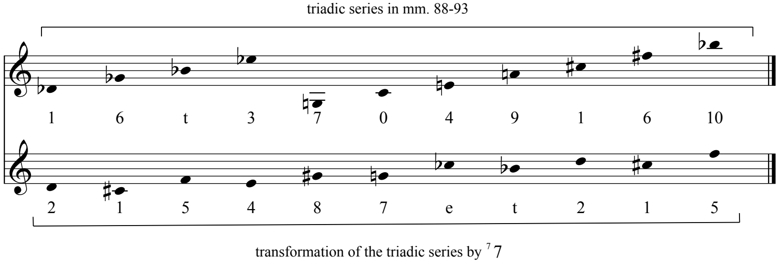
Octatonic set $\{0, 1, 3, 4, 6, 7, 9, 10\}$ contains major/minor chords

$$16t \xmapsto{R} 6t3 \xmapsto{P} t37 \xmapsto{R} 370 \xmapsto{P} 704 \xmapsto{R} 049 \xmapsto{P} 491 \xmapsto{R} 916$$

Consecutive chords have maximal overlap (well known neo-Riemannian operations)
Complete cycle
These major/minor chords are a maximal cover of the octatonic.

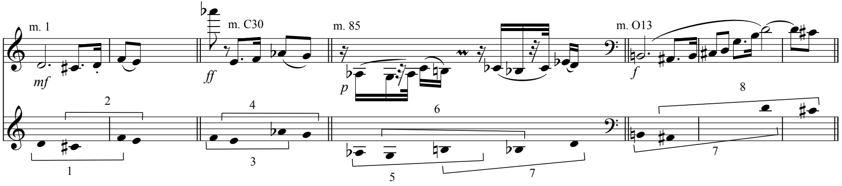
Certain Affine Image of Major/Minor *RP*-Cycle is Jet/Shark *RP*-Cycle

$^77 : \mathbb{Z}_{12} \to \mathbb{Z}_{12}$ denotes the affine map $x \mapsto 7x + 7$.
This maps the octatonic of mm 88 - 92 to the octatonic of the opening theme, and a *RP*-cycle to a "*RP*-cycle":



2154 is in opening theme: Jet 215 and Shark 154.
Note: Affine image of maximally overlapping chords also consists of maximally overlapping chords, cycle $\mapsto$ cycle, maximal cover $\mapsto$ maximal cover

Affine Image is Piece-wide Narrative Constructed from Opening Cell in Measures 1 and 2



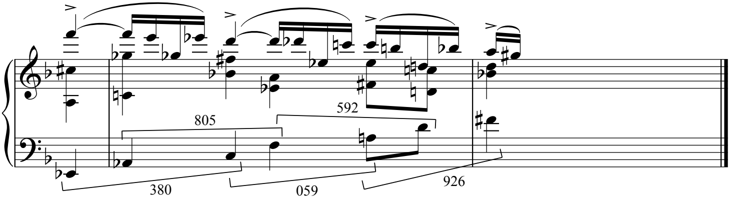
1 and 2: Opening violin motive

3 and 4: *Fortissimo* return of main theme, doubled in all four instruments

5,6, and 7: Just before major/minor *RP*-cycle, violin imitation

7 and 8: Return of opening theme in parallel major, with forms 7 and 8 as registral extremes

Further Instance of *RP*-Pattern within Third Octatonic, Measures 8 - 10



Third Octatonic $\{2,3,5,6,8,9,11,0\}$ contains major/minor chords

Consecutive chords have maximal overlap (well known neo-Riemannian operations)
However, pattern breaks off before attaining maximal cover.

Desiderata of a Mathematical Theory for this Analysis

To be of use in this analyis, a mathematical theory should:

1. Associate a neo-Riemannian-type group to 3-tuples (x_1, x_2, x_3) of pitch-classes, e.g. for (2,1,5) in opening theme
2. Have attendant theorems about duality with transposition, inversion, and affine maps, e.g. commutation with $^{7}7$
3. Show how to move between the associated neo-Riemannian type groups associated to *two* such 3-tuples (x_1, x_2, x_3) and (y_1, y_2, y_3) when related by an affine map, e.g. the major $(1, 6, 10)$ and the jet $(2, 1, 5)$ related by $^{7}7$
4. Show how to obtain substructures of neo-Riemannian type groups, e.g. for octatonic and its maximal major/minor cover or jet/shark cover
5. Show how to move between these substructures, e.g. between the three octatonics
6. Determine which subsets of the octatonic set generate maximal covers with simply transitive action under a neo-Riemannian type group action

The Neo-Riemannian *PLR*-Group

The Neo-Riemannian Transformation P

We consider three functions

$$P, L, R : S \to S.$$

Let $P(x)$ be that triad of opposite type as x with the first and third notes switched. For example

$$P\langle \mathbf{0}, 4, \mathbf{7}\rangle =$$

$$P(C\text{-major}) =$$

The set S of consonant triads	
Major Triads	Minor Triads
$C = \langle 0, 4, 7\rangle$	$\langle 0, 8, 5\rangle = f$
$C\sharp = D\flat = \langle 1, 5, 8\rangle$	$\langle 1, 9, 6\rangle = f\sharp = g\flat$
$D = \langle 2, 6, 9\rangle$	$\langle 2, 10, 7\rangle = g$
$D\sharp = E\flat = \langle 3, 7, 10\rangle$	$\langle 3, 11, 8\rangle = g\sharp = a\flat$
$E = \langle 4, 8, 11\rangle$	$\langle 4, 0, 9\rangle = a$
$F = \langle 5, 9, 0\rangle$	$\langle 5, 1, 10\rangle = a\sharp = b\flat$
$F\sharp = G\flat = \langle 6, 10, 1\rangle$	$\langle 6, 2, 11\rangle = b$
$G = \langle 7, 11, 2\rangle$	$\langle 7, 3, 0\rangle = c$
$G\sharp = A\flat = \langle 8, 0, 3\rangle$	$\langle 8, 4, 1\rangle = c\sharp = d\flat$
$A = \langle 9, 1, 4\rangle$	$\langle 9, 5, 2\rangle = d$
$A\sharp = B\flat = \langle 10, 2, 5\rangle$	$\langle 10, 6, 3\rangle = d\sharp = e\flat$
$B = \langle 11, 3, 6\rangle$	$\langle 11, 7, 4\rangle = e$

The Neo-Riemannian Transformation P

We consider three functions

$$P, L, R : S \to S.$$

Let $P(x)$ be that triad of opposite type as x with the first and third notes switched.
For example

$$P\langle \mathbf{0}, 4, \mathbf{7}\rangle = \langle \mathbf{7}, 3, \mathbf{0}\rangle$$

$$P(C\text{-major}) = c\text{-minor}$$

The set S of consonant triads	
Major Triads	Minor Triads
$C = \langle 0,4,7\rangle$	$\langle 0,8,5\rangle = f$
$C\sharp = D\flat = \langle 1,5,8\rangle$	$\langle 1,9,6\rangle = f\sharp = g\flat$
$D = \langle 2,6,9\rangle$	$\langle 2,10,7\rangle = g$
$D\sharp = E\flat = \langle 3,7,10\rangle$	$\langle 3,11,8\rangle = g\sharp = a\flat$
$E = \langle 4,8,11\rangle$	$\langle 4,0,9\rangle = a$
$F = \langle 5,9,0\rangle$	$\langle 5,1,10\rangle = a\sharp = b\flat$
$F\sharp = G\flat = \langle 6,10,1\rangle$	$\langle 6,2,11\rangle = b$
$G = \langle 7,11,2\rangle$	$\langle 7,3,0\rangle = c$
$G\sharp = A\flat = \langle 8,0,3\rangle$	$\langle 8,4,1\rangle = c\sharp = d\flat$
$A = \langle 9,1,4\rangle$	$\langle 9,5,2\rangle = d$
$A\sharp = B\flat = \langle 10,2,5\rangle$	$\langle 10,6,3\rangle = d\sharp = e\flat$
$B = \langle 11,3,6\rangle$	$\langle 11,7,4\rangle = e$

The Neo-Riemannian Transformations L and R

- Let $L(x)$ be that triad of opposite type as x with the second and third notes switched. For example

$$L\langle 0, \mathbf{4}, \mathbf{7}\rangle = \langle 11, \mathbf{7}, \mathbf{4}\rangle$$

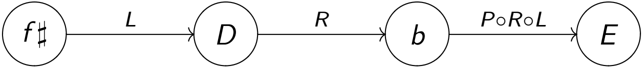
$$L(C\text{-major}) = e\text{-minor}.$$

- Let $R(x)$ be that triad of opposite type as x with the first and second notes switched. For example

$$R\langle \mathbf{0}, \mathbf{4}, 7\rangle = \langle \mathbf{4}, \mathbf{0}, 9\rangle$$

$$R(C\text{-major}) = a\text{-minor}.$$

Example: "Oh! Darling" from the Beatles



$E+$ A E
Oh____ Darling please believe me
$f\sharp$ D
I'll never do you no harm
$b7$ $E7$
Be-lieve me when I tell you
$b7$ $E7$ A
I'll never do you no harm

The Neo-Riemannian *PLR*-Group and Duality

Definition

The neo-Riemannian PLR-group is the subgroup of permutations of S generated by P,L, and R.

Theorem (Lewin 80's, Hook 2002, ...)

The PLR group is dihedral of order 24 and is generated by L and R.

Theorem (Lewin 80's, Hook 2002, ...)

The PLR group is dual to the T/I group in the sense that each is the centralizer of the other in the symmetric group on the set S of major and minor triads. Moreover, both groups act simply transitively on S.

Extension of Neo-Riemannian Theory

Theorem (Fiore–Satyendra, 2005)

Let $x_1, \ldots, x_n \in \mathbb{Z}_m$ *and suppose that there exist* x_q, x_r *in the list such that* $2(x_q - x_r) \neq 0$. *Let* S *be the family of* $2m$ *pitch-class segments that are obtained by transposing and inverting the pitch-class segment* $X = \langle x_1, \ldots, x_n \rangle$. *Then:*

1. *The transposition-inversion group acts simply transitively on* S *and has order* $2m$.
2. *Its centralizer is also dihedral of order* $2m$, *and is generated by any PLR-type operation and*

$$Q_1(Y) := \begin{cases} T_1 Y \text{ if } Y \text{ is a transposed form of } X \\ T_{-1} Y \text{ if } Y \text{ is an inverted form of } X. \end{cases}$$

Constructing Sub Dual Groups

Dual Groups

Definition (Dual groups in the sense of Lewin)

Let $\mathrm{Sym}(S)$ be the symmetric group on the set S. Two subgroups G and H of the symmetric group $\mathrm{Sym}(S)$ are called *dual* if their natural actions on S are simply transitive and each is the centralizer of the other, that is,

$$C_{\mathrm{Sym}(S)}(G) = H \quad \text{and} \quad C_{\mathrm{Sym}(S)}(H) = G.$$

Example

The T/I-group and the PLR-group are dual in the symmetric group on consonant triads.

Example (Cayley)

All dual groups arise as the left and right multiplication of a group on itself.

Constructing Sub Dual Groups

Theorem (Fiore–Noll, 2011)

Given a pair of dual groups, G and H acting on S, we may easily construct a subdual pair, G_0 and H_0 acting on S_0, as follows.

1. *Pick $G_0 \leq G$ and $s_0 \in S$.*
2. *Let $S_0 := G_0 s_0$ and $H_0 := \{h \in H \,|\, hs_0 \in S_0\}$.*
3. *Then G_0 and H_0 are dual groups acting on S_0.*
4. *If $k \in H$, and we transform S_0 to kS_0, then H_0 transforms to kH_0k^{-1}.*

Example: $\{C, c\}$

Example

$G = PLR\text{-group}$
$H = T/I\text{-group}$
$S = \text{consonant triads.}$

1. Pick $G_0 = \{Id, P\}$ and $s_0 = C$.
2. Then $S_0 = G_0 s_0 = \{C, c\}$
   H_0 consists of solutions to $T_i C = C$ and $I_j C = c$,
   $H_0 = \{Id, I_7\}$.
3. Then $\{Id, P\}$ and $\{Id, I_7\}$ are dual groups on $\{C, c\}$.
4. For any T_ℓ, $\{Id, P\}$ and $\{Id, T_\ell I_7 T_{-\ell}\}$ are dual groups on $\{T_\ell C, T_\ell c\}$.

Example: Hexatonic (Cohn 1996) and (Clampitt 1998)

Example

G = PLR-group, H = T/I-group, S = consonant triads.

1. *Pick $G_0 = \langle P, L\rangle$ and $s_0 = E\flat$.*
2. *Then $S_0 = G_0 s_0 = \{E\flat, e\flat, B, b, G, g\}$*
   $H_0 = \{Id, T_4, T_8, I_1, I_5, I_9\}$ consists of solutions to

$$T_i E\flat = E\flat \qquad I_\ell E\flat = e\flat$$
$$T_j E\flat = G \qquad I_m E\flat = g$$
$$T_k E\flat = B \qquad I_n E\flat = b.$$

3. *Then $\langle P, L\rangle$ and $\{Id, T_4, T_8, I_1, I_5, I_9\}$ are dual groups on $\{E\flat, e\flat, B, b, G, g\}$.*
4. *The transforms are the hyperhexatonic.*

Underlying set is $\{2, 3, 6, 7, 10, 11\}$.

Example: Octatonic

Example

G = PLR-group, H = T/I-group, S = consonant triads.

1. *Pick $G_0 = \langle R, P \rangle$ and $s_0 = G\flat$.*
2. *Then $S_0 = G_0 s_0 = \{G\flat, e\flat, E\flat, c, C, a, A, f\sharp\}$ (measures 88 - 92 of Schoenberg!)*
   $H_0 = \{Id, T_3, T_6, T_9, I_1, I_4, I_7, I_{10}\}$ by solving equations
3. *Then $\langle R, P \rangle$ and H_0 are dual groups on $\{G\flat, e\flat, E\flat, c, C, a, A, f\sharp\}$.*
4. *The transforms are the hyperoctatonic.*

Underlying set is octatonic $\{0, 1, 3, 4, 6, 7, 9, 10\}$.

Other Octatonic Covers

Proposition

Any 3-element subset X of the octatonic

$$\{0,1,3,4,6,7,9,10\} \subset \mathbb{Z}_{12}$$

generates a simply transitive cover with respect to the set-wise stabilizer $\{T_0, T_3, T_6, T_9, I_1, I_4, I_7, I_{10}\}$.

$$\begin{aligned}
\{0,4,7\} \times 1 &= \{0,4,7\} &&= \text{major type}\\
\{0,4,7\} \times 2 &= \{0,8,2\} &&= \text{strain type}\\
\{0,4,7\} \times 5 &= \{0,8,11\} &&= \text{shark type}\\
\{0,4,7\} \times 7 &= \{0,4,1\} &&= \text{jet type}\\
\{0,4,7\} \times 10 &= \{0,4,10\} &&= \text{stride type}\\
\{0,4,7\} \times 11 &= \{0,8,5\} &&= \text{minor type.}
\end{aligned}$$

Simply Transitive Groups for Other Octatonic Covers

Only multiples of major chord that have 3 notes:

$$\begin{array}{lcl}
\{0,4,7\} \times 1 & = & \{0,4,7\} & = & \text{major type} \\
\{0,4,7\} \times 2 & = & \{0,8,2\} & = & \text{strain type} \\
\{0,4,7\} \times 5 & = & \{0,8,11\} & = & \text{shark type} \\
\{0,4,7\} \times 7 & = & \{0,4,1\} & = & \text{jet type} \\
\{0,4,7\} \times 10 & = & \{0,4,10\} & = & \text{stride type} \\
\{0,4,7\} \times 11 & = & \{0,8,5\} & = & \text{minor type.}
\end{array}$$

All of these types occur in Schoenberg, String Quartet 1, Opus 7.

Fiore–Satyendra 2005 $\Rightarrow$ Each multiple has a neo-Riemannian type group associated to it.

Fiore–Noll 2011 $\Rightarrow$ Each multiple generates a simply transitive octatonic cover (take dual to set wise octatonic T/I stabilizer).

Morphisms of Simply Transitive Group Actions

Morphisms of Group Actions

Definition (Morphism of Group Actions)

Suppose (G_1, S_1) *and* (G_2, S_2) *are group actions. A* **morphism of group actions**

$$(f, \varphi)\colon (G_1, S_1) \longrightarrow (G_2, S_2)$$

consists of a function $f : S_1 \to S_2$ *and a group homomorphism* $\varphi\colon G_1 \to G_2$ *such that*

$$\begin{array}{ccc} S_1 & \xrightarrow{\;f\;} & S_2 \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \varphi(g)} \\ S_1 & \xrightarrow{\;f\;} & S_2 \end{array}$$

commutes for all $g \in G_1$.

Affine Maps are Morphisms for Groups Generated by P, L, and R-Analogues

Definition (Affine Map)

A function $f : \mathbb{Z}_{12} \to \mathbb{Z}_{12}$ is called **affine** *if there exist m and b such that $f(x) = mx + b$ for all $x \in \mathbb{Z}_{12}$.*

Proposition

Suppose $f : \mathbb{Z}_{12} \to \mathbb{Z}_{12}$ is affine. Then f commutes with P, L, and R-analogues, that is

$$Pf = fP \qquad Lf = fL \qquad Rf = fR.$$

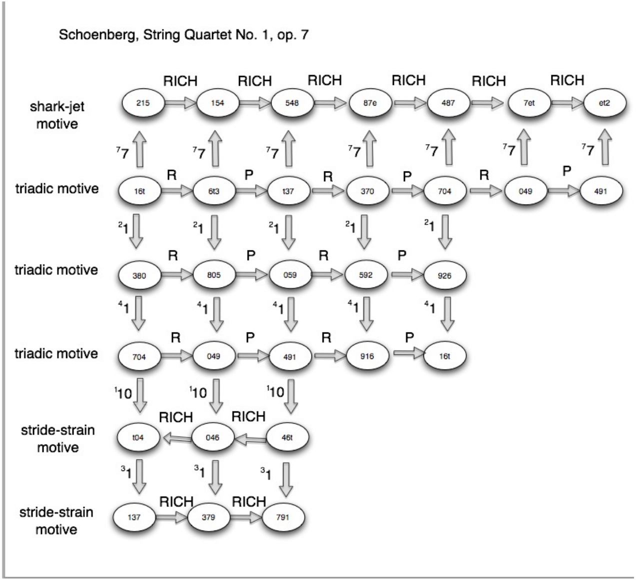
Example

$$\begin{array}{ccc} (2,1,5) & \xmapsto{R} & (1,5,4) \\ {}^{7}7 \uparrow & & {}^{7}7 \uparrow \\ (1,6,10) & \xmapsto{R} & (6,1,3) \end{array}$$

Return to Musical Analysis

The Affine Morphism $^{7}7$ Indicates a Parallel Organization Between the *RP* Major/Minor Cycle and the Jet-Shark Piece-wide Narrative.

Total Network



Total Network

Schoenberg, String Quartet No. 1, op. 7