What is Mathematical Music Theory? An Introduction via Perspectives on Consonant Triads

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What is Mathematical Music Theory?

Mathematical music theory uses *modern mathematical structures* to

1. analyze works of music (describe and explain them),
2. study, characterize, and reconstruct musical objects such as the consonant triad, the diatonic scale, the Ionian mode, the consonance/dissonance dichotomy...
3. compose
4. ...

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Levels of Musical Reality, Hugo Riemann

There is a distinction between three levels of musical reality.

- Physical level: a tone is a pressure wave moving through a medium, "*Ton*"
- Psychological level: a tone is our experience of sound, "*Tonempfindung*"
- Intellectual level: a tone is a position in a tonal system, described in a syntactical meta-language, "*Tonvorstellung*". Mathematical music theory belongs to this realm.

Work of Mazzola and Collaborators

- Mazzola, Guerino. *Gruppen und Kategorien in der Musik. Entwurf einer mathematischen Musiktheorie*. Research and Exposition in Mathematics, 10. Heldermann Verlag, Berlin, 1985.
- Mazzola, Guerino. *The topos of music. Geometric logic of concepts, theory, and performance*. In collaboration with Stefan Göller and Stefan Müller. Birkhäuser Verlag, Basel, 2002.
- Noll, Thomas, *Morphologische Grundlagen der abendländischen Harmonik* in: Moisei Boroda (ed.), *Musikometrika* 7, Bochum: Brockmeyer, 1997.

These developed a mathematical meta-language for music theory, investigated concrete music-theoretical questions, analyzed works of music, and did compositional experiments (Noll 2005).

Lewin

- Lewin, David. *Generalized Musical Intervals and Transformations*, Yale University Press, 1987.
- Lewin, David. *Musical Form and Transformation: 4 Analytic Essays*, Yale University Press, 1993.

Lewin introduced *generalized interval systems* to analyze works of music. The operations of Hugo Riemann were a point of departure. Mathematically, a generalized interval system is a *simply transitive group action.*

II. Basics

The $\mathbb{Z}_{12}$ Model of Pitch Class

We have a bijection between the set of pitch classes and $\mathbb{Z}_{12}$.

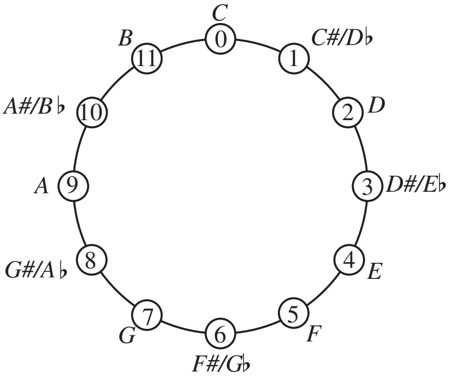

Figure: The musical clock.

Transposition and Inversion

We have the musical operations *transposition* (rotation)

$$T_n : \mathbb{Z}_{12} \to \mathbb{Z}_{12}$$

$$T_n(x) := x + n$$

and *inversion* (reflection)

$$I_n : \mathbb{Z}_{12} \to \mathbb{Z}_{12}$$

$$I_n(x) := -x + n.$$

These generate the dihedral group of order 24 (symmetries of 12-gon).

III. Characterization of the Consonant Triad

Major and Minor Triads

Major and minor triads are very common in Western music. (Audio)

$$C\text{-major} = \langle C, E, G \rangle = \langle 0, 4, 7 \rangle$$

$$c\text{-minor} = \langle G, E\flat, C \rangle = \langle 7, 3, 0 \rangle$$

The set S of consonant triads	
Major Triads	Minor Triads
$C = \langle 0,4,7 \rangle$	$\langle 0,8,5 \rangle = f$
$C\sharp = D\flat = \langle 1,5,8 \rangle$	$\langle 1,9,6 \rangle = f\sharp = g\flat$
$D = \langle 2,6,9 \rangle$	$\langle 2,10,7 \rangle = g$
$D\sharp = E\flat = \langle 3,7,10 \rangle$	$\langle 3,11,8 \rangle = g\sharp = a\flat$
$E = \langle 4,8,11 \rangle$	$\langle 4,0,9 \rangle = a$
$F = \langle 5,9,0 \rangle$	$\langle 5,1,10 \rangle = a\sharp = b\flat$
$F\sharp = G\flat = \langle 6,10,1 \rangle$	$\langle 6,2,11 \rangle = b$
$G = \langle 7,11,2 \rangle$	$\langle 7,3,0 \rangle = c$
$G\sharp = A\flat = \langle 8,0,3 \rangle$	$\langle 8,4,1 \rangle = c\sharp = d\flat$
$A = \langle 9,1,4 \rangle$	$\langle 9,5,2 \rangle = d$
$A\sharp = B\flat = \langle 10,2,5 \rangle$	$\langle 10,6,3 \rangle = d\sharp = e\flat$
$B = \langle 11,3,6 \rangle$	$\langle 11,7,4 \rangle = e$

Major and Minor Triads

The T/I-group acts on the set S of major and minor triads.

$$T_1\langle 0,4,7\rangle = \langle T_1 0, T_1 4, T_1 7\rangle = \langle 1,5,8\rangle$$

$$I_0\langle 0,4,7\rangle = \langle I_0 0, I_0 4, I_0 7\rangle = \langle 0,8,5\rangle$$

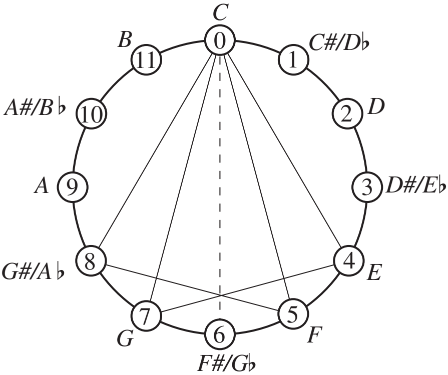

Figure: I_0 applied to a C-major triad yields an f-minor triad.

Neo-Riemannian Music Theory

- Recent work focuses on the neo-Riemannian operations P, L, and R.
- P, L, and R generate a dihedral group, called the *neo-Riemannian group*. As we'll see, this group is *dual* to the T/I group in the sense of Lewin.
- These transformations arose in the work of the 19th century music theorist Hugo Riemann, and have a pictorial description on the Oettingen/Riemann *Tonnetz*.
- P, L, and R are defined in terms of common tone preservation.

The neo-Riemannian Transformation P

We consider three functions

$$P, L, R : S \to S.$$

Let $P(x)$ be that triad of opposite type as x with the first and third notes switched.

For example

$P\langle \mathbf{0}, 4, \mathbf{7} \rangle =$

$P(C\text{-major}) =$

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$$P\langle \mathbf{0}, 4, \mathbf{7} \rangle = \langle \mathbf{7}, 3, \mathbf{0} \rangle$$

$$P(C\text{-major}) = c\text{-minor}$$

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The neo-Riemannian Transformations L and R

- Let $L(x)$ be that triad of opposite type as x with the second and third notes switched. For example

$$L\langle 0, \mathbf{4}, \mathbf{7}\rangle = \langle 11, \mathbf{7}, \mathbf{4}\rangle$$

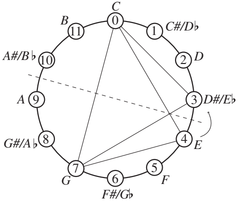
$$L(C\text{-major}) = e\text{-minor}.$$

- Let $R(x)$ be that triad of opposite type as x with the first and second notes switched. For example

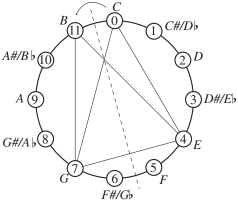
$$R\langle \mathbf{0}, \mathbf{4}, 7\rangle = \langle \mathbf{4}, \mathbf{0}, 9\rangle$$

$$R(C\text{-major}) = a\text{-minor}.$$

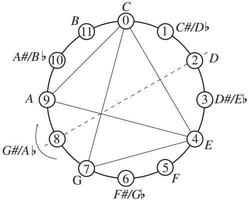
Minimal motion of the moving voice under P, L, and R.



$PC = c$



$LC = e$



$RC = a$

Uniqueness of the Consonant Triad

Theorem (Richard Cohn, 1997)

The consonant triad is uniquely characterized by parsimonious voice leading. Namely, suppose $\langle 0, x, y \rangle$ is a 3-note chord in $\mathbb{Z}_{12}$, and we generate its orbit under the dihedral group. Then the P, L, R operations associated to the orbit of $\langle 0, x, y \rangle$ admit parsimonious voice leading if and only if $\langle 0, x, y \rangle$ is a consonant triad.

Remarkable: this result is completely independent of the acoustic properties of the consonant triad!

Example: The Elvis Progression

The Elvis Progression I-VI-IV-V-I from 50's Rock is:

$$C \xrightarrow{R} a \xrightarrow{L} F \xrightarrow{R\circ L\circ R\circ L} G \xrightarrow{L\circ R} C$$

Example: “Oh! Darling” from the Beatles

$$f\sharp \xrightarrow{L} D \xrightarrow{R} b \xrightarrow{P\circ R\circ L} E$$

$E+$ A E
Oh____ Darling please believe me
$f\sharp$ D
I’ll never do you no harm
$b7$ $E7$
Be-lieve me when I tell you
$b7$ $E7$ A
I’ll never do you no harm

The neo-Riemannian *PLR*-Group and Duality

Definition

The neo-Riemannian PLR-group is the subgroup of permutations of S generated by P, L, and R.

Theorem (Lewin 80's, Hook 2002, ...)

The PLR group is dihedral of order 24 and is generated by L and R.

Theorem (Lewin 80's, Hook 2002, ...)

The PLR group is dual to the T/I group in the sense that each is the centralizer of the other in the symmetric group on the set S of major and minor triads. Moreover, both groups act simply transitively on S.

IV. Geometry of Consonant Triads

The *Tonnetz* Graph and its Dual are Tori

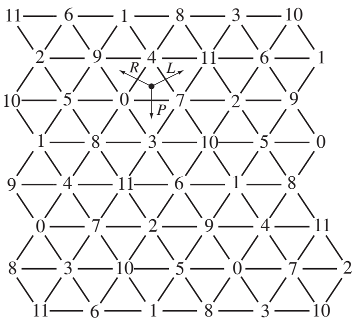


Figure: The *Tonnetz*.

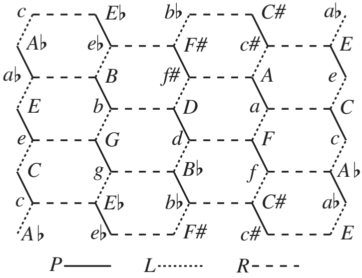


Figure: Douthett and Steinbach's Graph.

See Crans–Fiore–Satyendra, *Musical Actions of Dihedral Groups*, American Mathematical Monthly, 2009.

The Dual Graph to the *Tonnetz* on a Torus

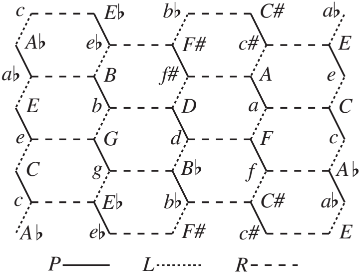


Figure: Douthett and Steinbach's Graph.

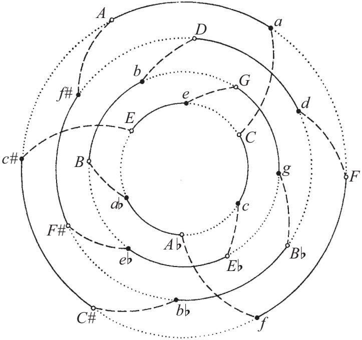
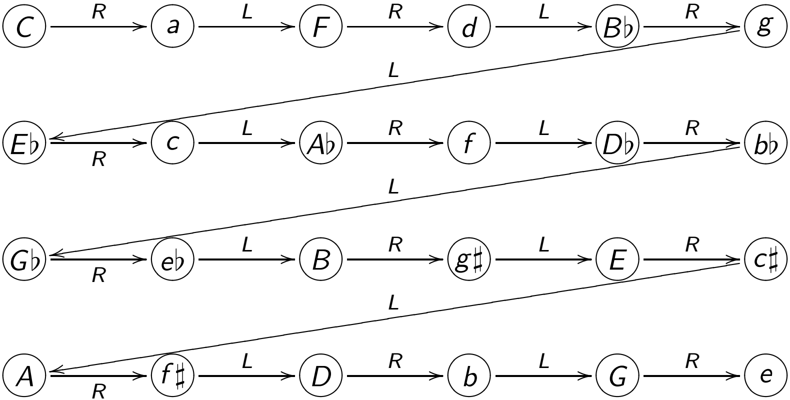


Figure: Waller's Torus.

Beethoven's 9th, 2nd Mvmt, Measures 143-175 (Cohn)



Thus far we have seen:

- How to encode pitch classes as integers modulo 12, and consonant triads as 3-tuples of integers modulo 12
- How the T/I-group acts componentwise on consonant triads
- How the PLR-group acts on consonant triads
- Duality between the T/I-group and the PLR-group
- Geometric depictions on the torus and musical examples.

But most music does not consist entirely of triads!
A mathematical perspective helps here.

V. Extension of Neo-Riemannian Theory

Extension of Neo-Riemannian Theory

Theorem (Fiore–Satyendra, 2005)

Let $x_1, \dots, x_n \in \mathbb{Z}_m$ and suppose that there exist x_q, x_r in the list such that $2(x_q - x_r) \neq 0$. Let S be the family of $2m$ pitch-class segments that are obtained by transposing and inverting the pitch-class segment $X = \langle x_1, \dots, x_n \rangle$. Then the $2m$ transpositions and inversions act simply transitively on S.

Extension of Neo-Riemannian Theory: Duality

Theorem (Fiore–Satyendra, 2005)

Fix $1 \leq k, \ell \leq n$. Define

$$K(Y) := I_{y_k + y_\ell}(Y)$$

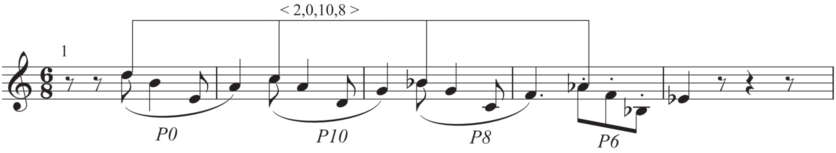
$$Q_i(Y) := \begin{cases} T_i Y \text{ if } Y \text{ is a transposed form of } X \\ T_{-i} Y \text{ if } Y \text{ is an inverted form of } X. \end{cases}$$

Then K and Q_1 generate the centralizer of the T/I group of order $2m$. This centralizer is called the generalized contextual group. It is dihedral of order $2m$, and its centralizer is the mod m T/I-group. Moreover, the generalized contextual group acts simply transitively on S.

Subject of Hindemith, *Ludus Tonalis*, Fugue in *E*

Let's see what this theorem can do for us in an analysis of Hindemith's Fugue in *E* from *Ludus Tonalis*.

Subject:



The Musical Space S in the Fugue in E

Consider the four-note **motive**

$$P_0 = \langle D, B, E, A\rangle$$
$$= \langle 2, 11, 4, 9\rangle.$$

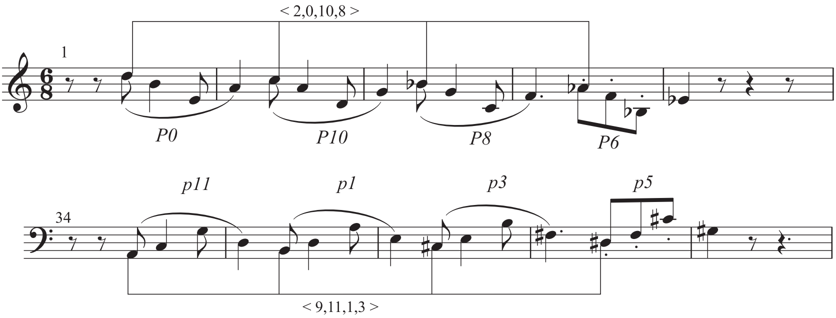
Transposed Forms

P_0	$\langle 2, 11, 4, 9\rangle$
P_1	$\langle 3, 0, 5, 10\rangle$
P_2	$\langle 4, 1, 6, 11\rangle$
P_3	$\langle 5, 2, 7, 0\rangle$
P_4	$\langle 6, 3, 8, 1\rangle$
P_5	$\langle 7, 4, 9, 2\rangle$
P_6	$\langle 8, 5, 10, 3\rangle$
P_7	$\langle 9, 6, 11, 4\rangle$
P_8	$\langle 10, 7, 0, 5\rangle$
P_9	$\langle 11, 8, 1, 6\rangle$
P_{10}	$\langle 0, 9, 2, 7\rangle$
P_{11}	$\langle 1, 10, 3, 8\rangle$

Inverted Forms

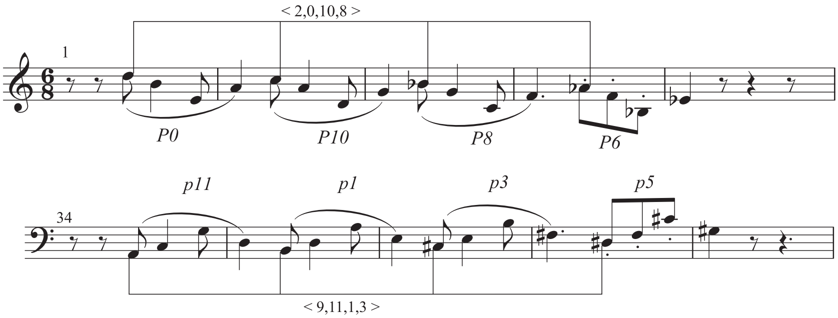
p_0	$\langle 10, 1, 8, 3\rangle$
p_1	$\langle 11, 2, 9, 4\rangle$
p_2	$\langle 0, 3, 10, 5\rangle$
p_3	$\langle 1, 4, 11, 6\rangle$
p_4	$\langle 2, 5, 0, 7\rangle$
p_5	$\langle 3, 6, 1, 8\rangle$
p_6	$\langle 4, 7, 2, 9\rangle$
p_7	$\langle 5, 8, 3, 10\rangle$
p_8	$\langle 6, 9, 4, 11\rangle$
p_9	$\langle 7, 10, 5, 0\rangle$
p_{10}	$\langle 8, 11, 6, 1\rangle$
p_{11}	$\langle 9, 0, 7, 2\rangle$

Q_{-2} Applied to Motive in Subject and I_{11}-Inversion



$$P_0 \xrightarrow{Q_{-2}} P_{10} \xrightarrow{Q_{-2}} P_8 \xrightarrow{Q_{-2}} P_6$$

Q_{-2} Applied to Motive in Subject and I_{11}-Inversion



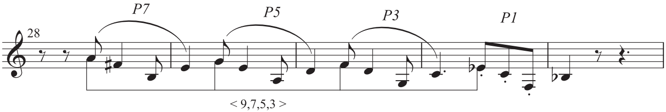
$$p_{11} \xrightarrow{Q_{-2}} p_1 \xrightarrow{Q_{-2}} p_3 \xrightarrow{Q_{-2}} p_5$$

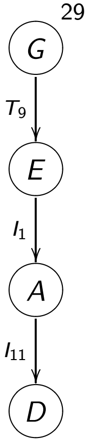
Product Network Encoding Subject and I_{11}-Inversion

$$
\begin{tikzcd}
\overset{\qquad 1}{P_0} \arrow[r, "Q_{-2}"] \arrow[d, "I_{11}"'] & P_{10} \arrow[r, "Q_{-2}"] \arrow[d, "I_{11}"'] & P_8 \arrow[r, "Q_{-2}"] \arrow[d, "I_{11}"'] & P_6 \arrow[d, "I_{11}"'] \\
\overset{\qquad 34}{p_{11}} \arrow[r, "Q_{-2}"] & p_1 \arrow[r, "Q_{-2}"] & p_3 \arrow[r, "Q_{-2}"] & p_5
\end{tikzcd}
$$

Our Theorem about duality guarantees this diagram commutes!

Self-Similarity: Local Picture





Self-Similarity: Global Picture and Local Picture

$$
\begin{tikzcd}
\text{Row 1}^{\text{8-11}} \arrow[d, "T_9"'] \\
\text{Row 2}^{\text{57-60}} \arrow[d, "I_1"'] \\
\text{Row 3}^{\text{34-37}} \arrow[d, "I_{11}"'] \\
\text{Row 4}^{\text{1-4}}
\end{tikzcd}
\qquad
\begin{tikzcd}
P_5 \arrow[r, "Q_{-2}"] \arrow[d, "T_9"'] & P_3 \arrow[r, "Q_{-2}"] \arrow[d, "T_9"'] & P_1 \arrow[r, "Q_{-2}"] \arrow[d, "T_9"'] & P_{11} \arrow[d, "T_9"'] \\
P_2 \arrow[r, "Q_{-2}"] \arrow[d, "I_1"'] & P_0 \arrow[r, "Q_{-2}"] \arrow[d, "I_1"'] & P_{10} \arrow[r, "Q_{-2}"] \arrow[d, "I_1"'] & P_8 \arrow[d, "I_1"'] \\
p_{11} \arrow[r, "Q_{-2}"] \arrow[d, "I_{11}"'] & p_1 \arrow[r, "Q_{-2}"] \arrow[d, "I_{11}"'] & p_3 \arrow[r, "Q_{-2}"] \arrow[d, "I_{11}"'] & p_5 \arrow[d, "I_{11}"'] \\
P_0 \arrow[r, "Q_{-2}"] & P_{10} \arrow[r, "Q_{-2}"] & P_8 \arrow[r, "Q_{-2}"] & P_6
\end{tikzcd}
$$

Audio: Hindemith, *Ludus Tonalis*, Fugue in E

VI. The Topos of Triads and the PLR-Group (Fiore–Noll 2011)