What is Mathematical Music Theory?

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Introduction

Mathematical music theory uses *modern mathematical structures* to

1. analyze works of music (describe and explain them),
2. study, characterize, and reconstruct musical objects such as the consonant triad, the diatonic scale, the Ionian mode, the consonance/dissonance dichotomy...
3. compose
4. ...

Levels of Musical Reality, Hugo Riemann

There is a distinction between three levels of musical reality.

- Physical level: a tone is a pressure wave moving through a medium, "*Ton*"
- Psychological level: a tone is our experience of sound, "*Tonempfindung*"
- Intellectual level: a tone is a position in a tonal system, described in a syntactical meta-language, "*Tonvorstellung*". Mathematical music theory belongs to this realm.

Work of Mazzola and Collaborators

- Mazzola, Guerino. *Gruppen und Kategorien in der Musik. Entwurf einer mathematischen Musiktheorie*. Research and Exposition in Mathematics, 10. Heldermann Verlag, Berlin, 1985.
- Mazzola, Guerino. *The topos of music. Geometric logic of concepts, theory, and performance*. In collaboration with Stefan Göller and Stefan Müller. Birkhäuser Verlag, Basel, 2002.
- Noll, Thomas, *Morphologische Grundlagen der abendländischen Harmonik* in: Moisei Boroda (ed.), *Musikometrika* 7, Bochum: Brockmeyer, 1997.

These developed a mathematical meta-language for music theory, investigated concrete music-theoretical questions, analyzed works of music, and did compositional experiments (Noll 2005).

Lewin

- Lewin, David. *Generalized Musical Intervals and Transformations*, Yale University Press, 1987.
- Lewin, David. *Musical Form and Transformation: 4 Analytic Essays*, Yale University Press, 1993.

Lewin introduced *generalized interval systems* to analyze works of music. The operations of Hugo Riemann were a point of departure. Mathematically, a generalized interval system is a *simply transitive group action.*

II. Basics

The $\mathbb{Z}_{12}$ Model of Pitch Class

We have a bijection between the set of pitch classes and $\mathbb{Z}_{12}$.

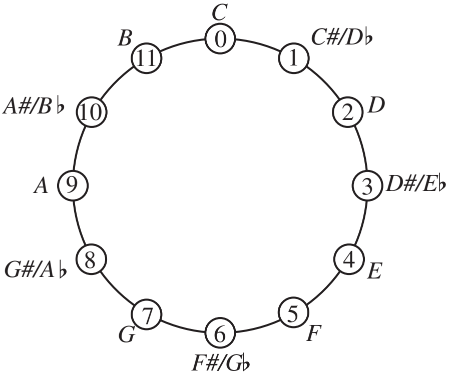

Figure: The musical clock.

Transposition and Inversion

We have the musical operations *transposition* (rotation)

$$T_n : \mathbb{Z}_{12} \to \mathbb{Z}_{12}$$

$$T_n(x) := x + n$$

and *inversion* (reflection)

$$I_n : \mathbb{Z}_{12} \to \mathbb{Z}_{12}$$

$$I_n(x) := -x + n.$$

These generate the dihedral group of order 24 (symmetries of 12-gon).

III. Characterization of the Consonant Triad

Major and Minor Triads

Major and minor triads are very common in Western music.

$$C\text{-major} = \langle C, E, G\rangle = \langle 0, 4, 7\rangle$$

$$c\text{-minor} = \langle G, E\flat, C\rangle = \langle 7, 3, 0\rangle$$

The set S of consonant triads	
Major Triads	Minor Triads
$C = \langle 0, 4, 7\rangle$	$\langle 0, 8, 5\rangle = f$
$C\sharp = D\flat = \langle 1, 5, 8\rangle$	$\langle 1, 9, 6\rangle = f\sharp = g\flat$
$D = \langle 2, 6, 9\rangle$	$\langle 2, 10, 7\rangle = g$
$D\sharp = E\flat = \langle 3, 7, 10\rangle$	$\langle 3, 11, 8\rangle = g\sharp = a\flat$
$E = \langle 4, 8, 11\rangle$	$\langle 4, 0, 9\rangle = a$
$F = \langle 5, 9, 0\rangle$	$\langle 5, 1, 10\rangle = a\sharp = b\flat$
$F\sharp = G\flat = \langle 6, 10, 1\rangle$	$\langle 6, 2, 11\rangle = b$
$G = \langle 7, 11, 2\rangle$	$\langle 7, 3, 0\rangle = c$
$G\sharp = A\flat = \langle 8, 0, 3\rangle$	$\langle 8, 4, 1\rangle = c\sharp = d\flat$
$A = \langle 9, 1, 4\rangle$	$\langle 9, 5, 2\rangle = d$
$A\sharp = B\flat = \langle 10, 2, 5\rangle$	$\langle 10, 6, 3\rangle = d\sharp = e\flat$
$B = \langle 11, 3, 6\rangle$	$\langle 11, 7, 4\rangle = e$

Neo-Riemannian P, L, R operations

Each consonant triad shares two common tones with three other consonant triads:

$$P\langle \mathbf{0}, 4, \mathbf{7} \rangle = \langle \mathbf{7}, 3, \mathbf{0} \rangle$$
$$P(C\text{-major}) = c\text{-minor}$$

$$L\langle 0, \mathbf{4}, \mathbf{7} \rangle = \langle 11, \mathbf{7}, \mathbf{4} \rangle$$
$$(C\text{-major}) = e\text{-minor}$$

$$R\langle \mathbf{0}, \mathbf{4}, 7 \rangle = \langle \mathbf{4}, \mathbf{0}, 9 \rangle$$
$$R(C\text{-major}) = a\text{-minor}.$$

Parsimonious voice leading: minimal motion in moving voice!

Uniqueness of the Consonant Triad

Theorem (Richard Cohn, 1997)

The consonant triad is uniquely characterized by parsimonious voice leading. Namely, suppose $\langle 0, x, y \rangle$ *is a 3-note chord in* $\mathbb{Z}_{12}$*, and we generate its orbit under the dihedral group. Then the* P*,* L*,* R *operations associated to the orbit of* $\langle 0, x, y \rangle$ *admit parsimonious voice leading if and only if* $\langle 0, x, y \rangle$ *is a consonant triad.*

Remarkable: this result is completely independent of the acoustic properties of the consonant triad!

IV. Characterization of the Diatonic Scale

Scales

Definition

A **scale** *is a subset of* $\mathbb{Z}_c$ *for some nonnegative integer* c*.*

Example

The **diatonic scale** *is*

$$\{0, 2, 4, 5, 7, 9, 11\} = \{C, D, E, F, G, A, B\}.$$

It is **generated** *by the perfect fifth=7, namely:*

F	*C*	*G*	*D*	*A*	*E*	*B*
5	*5+7*	*5+2*7*	*5 + 3*7*	*5+4*7*	*5+5*7*	*5+6*7*
5	*0*	*7*	*2*	*9*	*4*	*11.*

Uniqueness of the Diatonic Scale Theorem of Clough–Engebretsen–Kochavi, 1999

Suppose $S \subseteq \mathbb{Z}_c$ is a scale and satisfies the following properties.

1. S is **generated**.
2. S is **well-formed**.
3. S is **Myhill**.
4. S is **distributionally even**.
5. S is **maximally even**.
6. S is **deep**.
7. S is **hyperdiatonic**, that is, S is maximally even, $c = 2(|S| - 1)$, and $4|c$.
8. S is **Balzano**, that is, S is maximally even, $c = k(k+1)$, and S=generator=$2k+1$ with $k \geq 3$.

Then $c = 12$ and S is a translate of the diatonic scale in the previous example.

V. Characterization of the Ionian Mode

Idea of Noll's Characterization

Question: on which note does the diatonic scale begin?

Answer: Noll characterizes the Ionian mode (=major scale) in terms of step-interval patterns, fifth/fourth folding patterns, and division. (See page 88=10).

Words and Christoffel Words

Let $\{a, b\}^*$ denote the free monoid on the set $\{a, b\}$. Elements of $\{a, b\}^*$ are called **words**, e.g. $\emptyset$, ab, ba, and $abba$, are words. $a^{-1}b$ is not a word.

Definition

Let p and q be relatively prime integers. The **Christoffel word** w **of slope** $\frac{p}{q}$ **and length** $n = p + q$ *is the lower discretization of the line* $y = \frac{p}{q}x$. *That is*

$$w_i = \begin{cases} a & \textit{if } p \cdot i \textit{ mod } n > p \cdot (i-1) \textit{ mod } n \\ b & \textit{if } p \cdot i \textit{ mod } n < p \cdot (i-1) \textit{ mod } n. \end{cases}$$

Example

Tetractys.

Sturmian Morphisms

Consider the monoid homomorphisms $\{a, b\}^* \to \{a, b\}^*$ defined on generators a and b as follows.

	a	b
G	a	ab
$\tilde{G}$	a	ba
D	ba	b
$\tilde{D}$	ab	b
E	b	a

St_0 = monoid generated by G, $\tilde{G}$, D, $\tilde{D}$

= monoid of **special Sturmian morphisms**

St = monoid generated by G, $\tilde{G}$, D, $\tilde{D}$, and E

= monoid of **Sturmian morphisms**

Conjugates and Dividers

Two words w and w' are **conjugate** if there exist words u and v such that $w = uv$ and $w' = vu$, that is, w' is a **rotation** of w.

Theorem

A word r is conjugate to a Christoffel word if and only if $r = F(ab)$ for some Sturmian morphism $F \in St$.

If $r = F(ab)$ for a special Sturmian morphism $F \in St_0$, we factor r by writing $r = F(a)|F(b)$. The vertical line is the **divider**.

Christoffel Duality and its Extension to Conjugates

Definition

Let p and q be relatively prime. The **Christoffel dual word** w^* **to the Christoffel word** w **of slope** $\frac{p}{q}$ *is the Christoffel word of slope* $\frac{p*}{q^*}$ *where* p^* *and* q^* *are the multiplicative inverses of* p *and* q *in* $\mathbb{Z}_n$.

Example

The dual of $w = aab$ *is* $w^* = xxy$.

Example

The dual of the step interval pattern of the Lydian mode is the fifth-fourth folding pattern of the Lydian mode!

Christoffel Duality and its Extension to Conjugates

Theorem (Clampitt–Domínguez–Noll, 2007)

The assignment $w \mapsto w^*$ on Christoffel words extends to a bijection

$$\{\textit{conjugates of } w\} \xrightarrow{(-)^\square} \{\textit{conjugates of } w^*\}$$

called **plain adjoint**. *If F is a composite of G's and D's and $r = F(ab)$, then $r^\square = F^{rev}(xy)$ where F^{rev} is the reverse of F. A similar statement holds if F is a composite of G's and $\tilde{D}$'s.*

Example.
$GG\tilde{D}(ab) = aaabaab = r =$ step-interval pattern of Lydian
$\tilde{D}GG(xy) = \tilde{D}G(xxy) = \tilde{D}(xxxy) = xyxyxyy = r^\square =$
fifth-fourth folding pattern of Lydian

Characterization of Ionian Mode

Remarkable Fact: If we take in the Theorem

$$w = aaabaab = \text{ Lydian step-interval pattern},$$

then $(-)^{\square}$ is precisely the bijection between the seven diatonic modes and their fifth-fourth foldings!
See page 88=10 and page 94=16 of Noll.

Theorem (Divider Incidence, Clampitt–Domínguez–Noll, 2007)

Let r be a conjugate of a Christoffel word. The following are equivalent.

- $r = F(ab)$ *where* F *is a composite of* G*'s and* D*'s.*
- *The dividing note of* r *and* $r^{\square}$ *is the same.*

Thus, the Ionian mode is distinguished amongst all seven modes.
See page 96=18 of Noll for more characterizations.

VI. Summary

Society, Meetings, Journal

Professional Society: Society for Mathematics and Computation in Music

Journal: Journal of Mathematics and Music, promotional copies available

Meeting: Special Session on Mathematical Techniques in Musical Analysis, Joint Meeting, New Orleans, January 6th, 2011

Meeting: MAA Minicourse on Geometry and Algebra in Music Theory, Joint Meeting, New Orleans, January 7th and 9th, 2011

Meeting: MCM 2011, IRCAM, Paris, France, June 15-17, 2011