MAA Minicourse #3: Geometry and Algebra in Mathematical Music Theory
Part B, Classroom Presentation II: Generalized Commuting Groups

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THE GROUP-THEORETICAL APPROACH

1.1 Interval preservation under conjugation as a result of commutativity

$$\begin{aligned} \alpha^\beta &= (\beta\alpha)\beta^{-1} \\ &= (\alpha\beta)\beta^{-1} && \text{(by the commutative property)} \\ &= \alpha(\beta\beta^{-1}) && \text{(by the associative property)} \\ &= \alpha && \text{(by cancellation)} \end{aligned}$$

1.2 Centralizer of H in G

$$C_G H = \{\alpha \in G \mid \alpha\beta = \beta\alpha, \text{ for all } \beta \in H\}$$

1.3 Normalizer of H in G

$$N_G H = \{\alpha \in G \mid H^\alpha = H\}$$

1.4 Center of H

$$Z(H) = \{\alpha \in H \mid \alpha\beta = \beta\alpha, \text{ for all } \beta \in H\}$$

1.5 Point stabilizer of $x \in S$ in H

$$H_x = \{\alpha \in H \mid \alpha(x) = x\}$$

1.6 Transitive action

For any $x, y \in S$, there exists $\alpha \in H$, such that $\alpha(x) = y$.

1.7 Structure of a centralizer of a group H with a transitive action

$$C_{Sym(S)} H \cong N_H H_x / H_x$$

1.8 Semiregular action

$$H_x = 1, \text{ for all } x \in S$$

1.9 Structure of a centralizer of a group H with a simply transitive action

$$C_{Sym(S)} H \cong N_H H_x / H_x = H/1 = H$$

1.10 Orbit restriction

Let

$$\pi : H \to Sym(S)$$

be a permutation representation of H, where $\pi(h) = h^*$. Next, let $P \subseteq S$ be a union of some number of H-orbits in S. Given that H has an action on P, we may define a function

$$h^*|_P : P \to P$$

on P that agrees with h^*, which we call the restriction of h to P. Then, we may define the representation map

$$\pi|_P : H \to Sym(P),$$

where $\pi|_P(h) = h^*|_P$, for all $h \in H$. In this way, we may discuss the restriction of H to any (union) of its orbits.

1.11 Diagonal subgroup

D is a diagonal subgroup of G iff

1) for any $\alpha(R_1) \in D(R_1)$, there exists a unique $\alpha(R_i) \in D(R_i)$ for each R_i in the set of n orbits, such that $\alpha(R_1) \cdots \alpha(R_n) \in D$.
2) $D(R_1)$ is permutation isomorphic to $D(R_i)$ for all $R_i \in R$.

1.12 Wreath product $W = L \text{ wr}_\pi F$

1) W is a semidirect product of B by F where $B = L_1 \times \ldots \times L_n$ is a direct product of n copies of L.
2) F permutes $Q = \{L_i : 1 \le i \le n\}$ via conjugation, and the permutation representation of F on Q is equivalent to π.

1.13 Structure of a centralizer for a group D with a diagonal action

$$C_{Sym(S)}D = C_{\mathrm{Sym}(R_i)}D(\mathrm{R}_i) \wr Sym(R)$$

1.14 Structure of a centralizer of a single orbit restriction for a group D with a semiregular intransitive action

$$C_{\mathrm{Sym}(R_i)}D(\mathrm{R}_i) \cong D$$

1.15 Maximally embedded diagonal subgroup

Let P_j be a union of D-orbits. $D(P_j)$ is a maximally embedded diagonal subgroup of G iff m is the greatest possible number of orbits R_i satisfying the diagonal subgroup condition for $R_i \subseteq P_j$.

1.16 Structure of a centralizer for a group D with a nonsemiregular intransitive action

$$C_{Sym(S)}D = C_{Sym(P_1)}D(P_1) \times \ldots \times C_{Sym(P_n)}D(P_n).$$

Table. Partition of the universe of non-empty pitch-class sets into unions of set-classes over which the action of *T/I* has a maximally diagonal embedding

Label	(x, y) = number of T_n and I_n operators that hold a member of P_j invariant; I_n parity	Representative inclusive set-class	Number of set-classes
P_1	(1, 0)	[0, 3, 7]	127
P_2	(1, 1) even parity for I_n index	[0]	56
P_3	(1, 1) odd parity for I_n index	[0, 1]	25
P_4	(2, 0)	[0, 1, 3, 6, 7, 9]	1
P_5	(2, 2) even parity for I_n indices	[0, 2, 6, 8]	5
P_6	(2, 2) odd parity for I_n indices	[0, 1, 6, 7]	2
P_7	(3, 3) even parity for I_n indices	[0, 4, 8]	2
P_8	(3, 3) odd parity for I_n indices	[0, 1, 4, 5, 8, 9]	1
P_9	(4, 4) even and odd parity for I_n indices	[0, 3, 6, 9]	2
P_{10}	(6, 6) even parity for I_n indices	[0, 2, 4, 6, 8, 10]	1
P_{11}	(12, 12) even and odd parity for I_n indices	[0, 1, 2, …, 11]	1

1.17.1 Structure of $C_{Sym(S)}T/I$, where S = {universe of non-empty pcsets}

$$(D_{24}\ wr\ S_{127}) \times (C_2\ wr\ S_{56}) \times (C_2\ wr\ S_{25}) \times (D_{12}\ wr\ S_1) \times (C_2\ wr\ S_5) \times (C_2\ wr\ S_2) \times (C_2\ wr\ S_2) \times (C_2\ wr\ S_1) \times (1\ wr\ S_2) \times (C_2\ wr\ S_1) \times (1\ wr\ S_1)$$

1.17.2 Size of $C_{Sym(S)}T/I$, where S = {universe of non-empty pcsets}

$$(24^{127} \cdot 127!) \cdot (2^{56} \cdot 56!) \cdot (2^{25} \cdot 25!) \cdot (12^1 \cdot 1!) \cdot (2^5 \cdot 5!) \cdot (2^2 \cdot 2!) \cdot (2^2 \cdot 2!) \cdot (2^1 \cdot 1!) \cdot (1^2 \cdot 2!) \cdot (2^1 \cdot 1!) \cdot (1^1 \cdot 1!)$$

THE GRAPH-THEORETICAL APPROACH

Figure 1. Arrow preservation for network N_1 (“book diagram”)

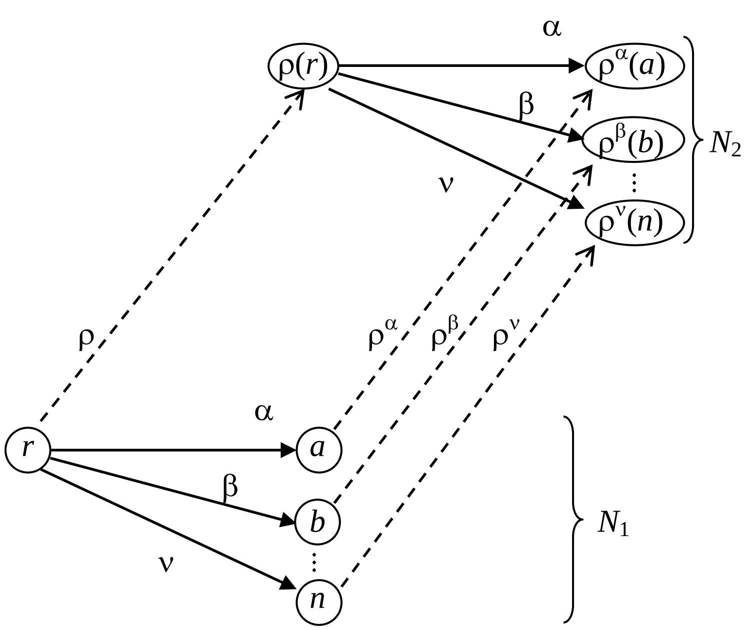


Figure 2. Arrow preservation in a *T/I* network

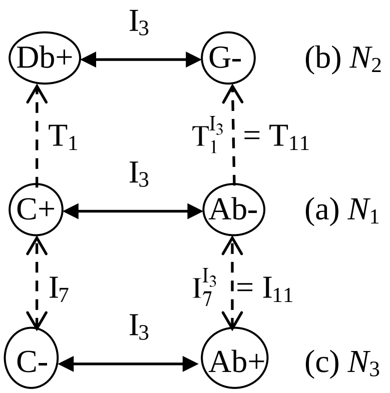
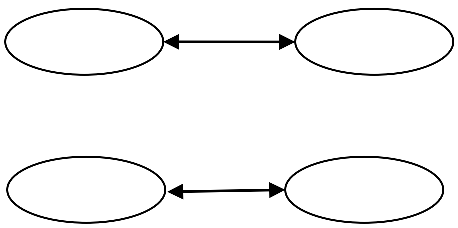


Figure 3. An unconnected graph



2.1 Number of networks that preserve arrow labels for groups G with semiregular actions
(n = number of orbits, m = number of connected components; p = size of orbit)

$$v = (n!/(n - m)!) \cdot p^m \quad \text{(visible on the network)}$$

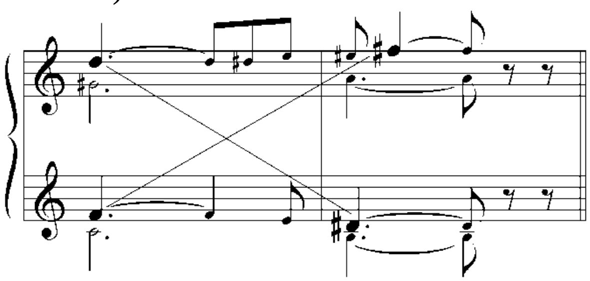
Example 1. Voice exchanges (and quasi voice exchanges) in *Tristan* Prelude

a) mm. 2-3 b) mm. 6-7

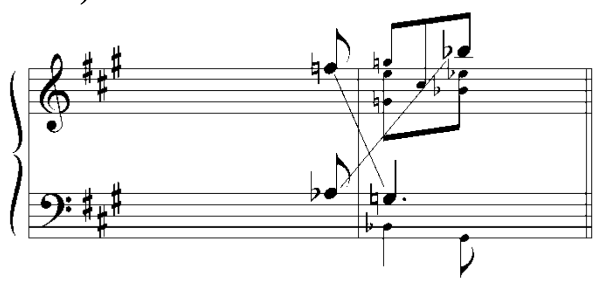
(*cont.*)

Example 1. Voice exchanges (and quasi voice exchanges) in *Tristan* Prelude, (*cont.*)

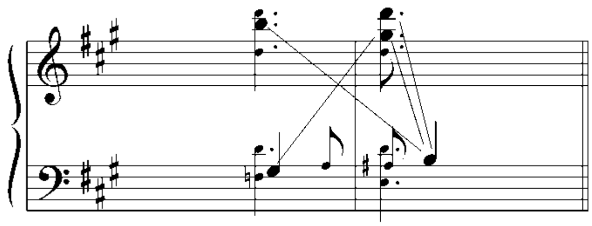
c) mm. 10-11



d) mm. 60-61



e) mm. 66-67



2.2.1 $S = (3\mathbb{Z}_{12} + 2) \times \mathbb{Z}_{12}$

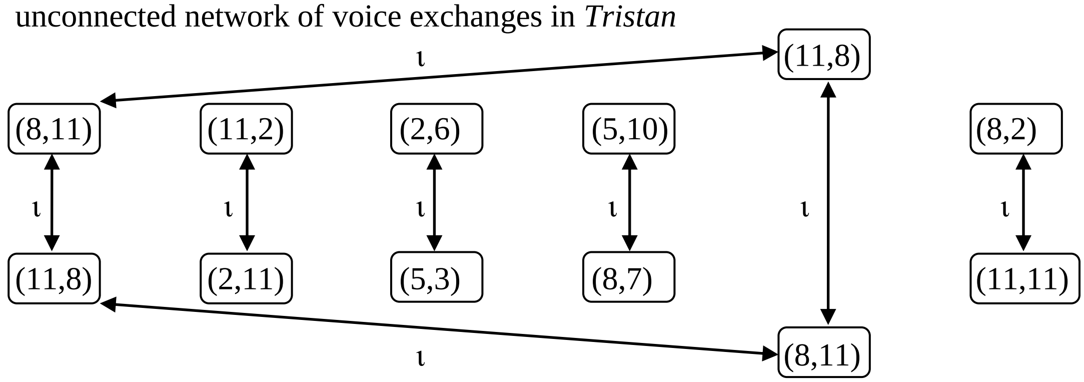
2.2.2 $\iota : (a, b) \in S \rightarrow (b - (b - a + 3 \text{ mod } 6), a + (b - a + 3 \text{ mod } 6))$

2.2.3 $T_3 : (a, b) \in S \rightarrow (a + 3, b + 3)$

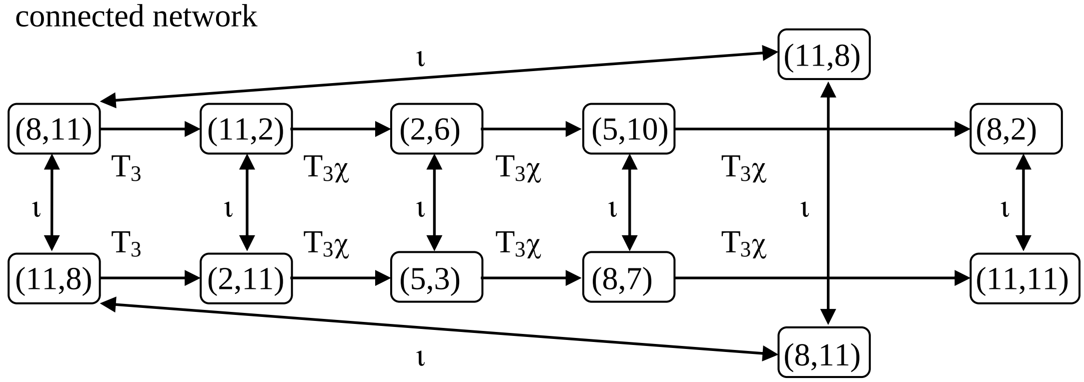
2.2.4 $\chi : (a, b) \in S \rightarrow (a, b + 1)$

Figure 4. Arrow preservation in intransitive semiregular networks

a) unconnected network of voice exchanges in *Tristan*

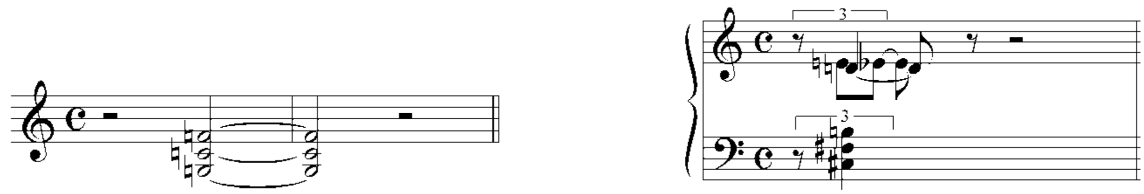


b) connected network



Example 2. Three trichords from Schoenberg's Op. 19, No. 6

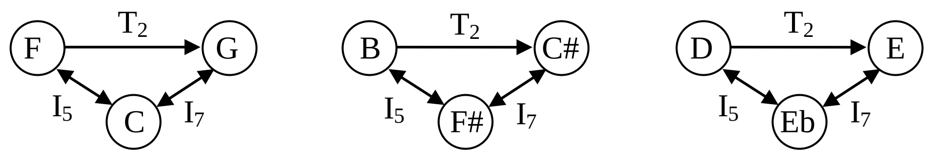
c)



a) b)

Figure 5. Three strongly isographic K-nets for Example 2 (from Lambert's 2002 analysis)

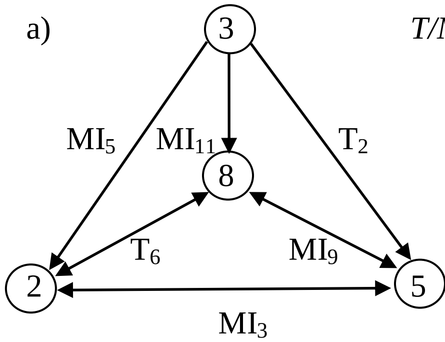
a) b) c)



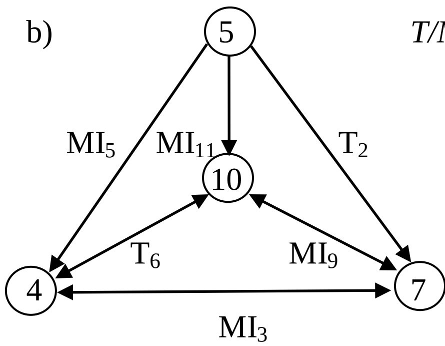
Example 3. Ran, String Quartet No. 1, Mov. II, mm. 121-125

Figure 6. *T/MI*-inclusive networks for Examples 3 and 4

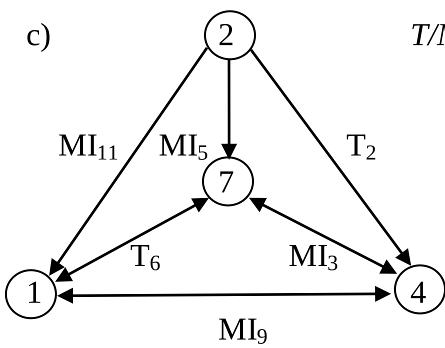
a) *T/MI*-inclusive network for m. 121



b) *T/MI*-inclusive network for m. 124



c) *T/MI*-inclusive network for m. 3



Example 4. Ran, String Quartet No. 1, Mov. II, mm. 1-3

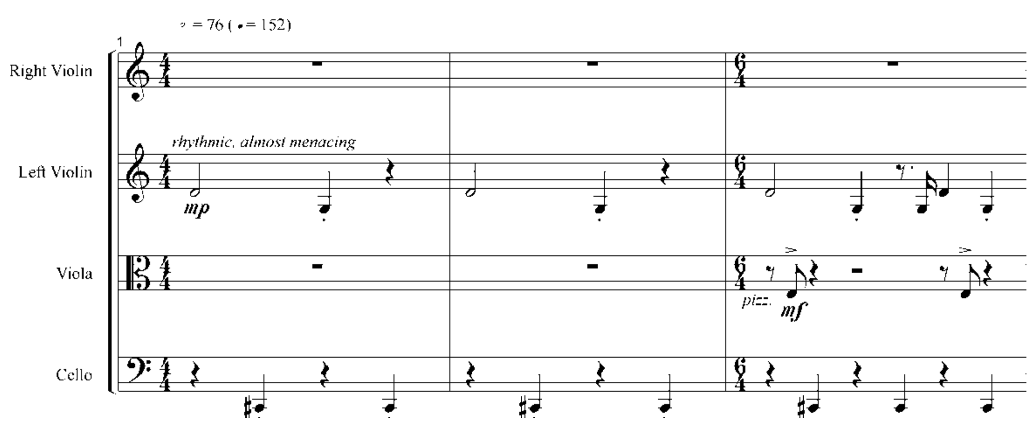
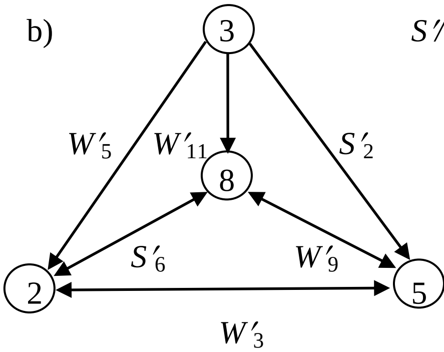


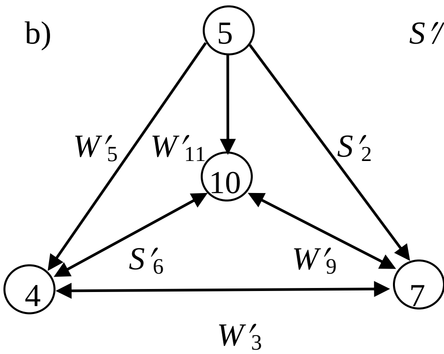
Figure 7. *S′/W′* networks with arrow preservation

b)



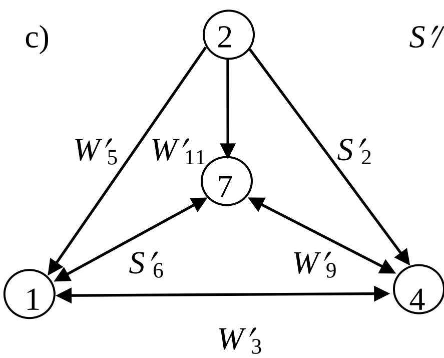
S′/W′-inclusive network for m. 121

b)



S′/W′-inclusive network for m. 124

c)



S′/W′-inclusive network for m. 3

SAMPLE HOMEWORK PROBLEMS

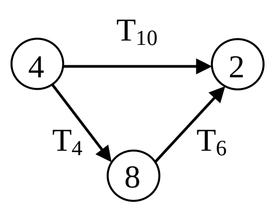
1) Determine generators for the commuting group in the symmetric group on the set of integers mod 6 for the group generated by (0,2,4)(1,3,5). How does the fact of this group's being abelian impact the commuting group?

2) Determine the size and structure of the commuting group in symmetric group on 12 pitch-classes for the group generated by the inversion operator I_1 := (0,1)(2,11)(3,10)(4,9)(5,8)(6,7). How does this structure contrast with the commuting group (in the same symmetric group) for the group generated by the inversion operator I_0 := (1,11)(2,10)(3,9)(4,8)(5,7)?

3) What is the kernel of the action of the (dihedral) musical transposition (translation) and inversion group's action on the set class of octatonic collections (i.e., all pitch-class sets that are translations and translated reflections of the pitch-class set {0,1,3,4,6,7,9,10})? How does this kernel function in determining the commuting group for the action of the musical transposition and inversion group on this set class?

4) Provide generators as operations on pitch-classes for a group whose commuting group is isomorphic to the wreath product $2^2 \wr S_3$ (Klein four-group by symmetric group of degree 3).

5) How many networks with nodes populated by pitch-classes have the same arrow labels as the network below? Is this the same as the size of the commuting group in in symmetric group on 12 pitch-classes for the musical transposition (translation) group?



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