Transformational Theory: Overview Presentation

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Music Theory in the words of David Lewin

"THEORY, then, attempts to describe the ways in which, given a certain body of literature, composers and listeners appear to have accepted sound as conceptually structured, categorically prior to one specific piece. E.g. one supposes that when Beethoven wrote, say, the *Eroica*, he had "in his ear" a "sound-universe" comprising his apperceptions of such abstractions as triad, scale, key, tonic, dominant, metric stress, etc. When he was composing the work, sounds did not present themselves to his imagination *solely* within the context of the piece itself, but also in the context of the sound-universe, as a general "way of hearing." Likewise, his listeners heard the work (as do we) not only in its own context, but also against this general background. ... it is with the structure of such general sound-universes that theory is concerned"

Lewin, David. "Behind the Beyond: A Response to Edward T. Cone." *Perspectives of New Music* 7, no. 2: 59-69. 1969.

Mathematical Music Theory

Mathematical Music Theory uses mathematics as a language to do Music Theory.

But not only that, Mathematical Music Theory has *theorems* about musical objects.

Mathematics also provides inspiration for new theorems in Mathematical Music Theory.
For example, see work of Noll–Clampitt–Domínguez inspired by algebraic combinatorics of words.

Introduction

Mathematical music theory uses *modern mathematical structures* to

1. analyze works of music (describe and explain them),
2. study, characterize, and reconstruct musical objects such as the consonant triad, the diatonic scale, the Ionian mode, the consonance/dissonance dichotomy...
3. compose
4. ...

Levels of Musical Reality, Hugo Riemann

There is a distinction between three levels of musical reality.

- Physical level: a tone is a pressure wave moving through a medium, "*Ton*"
- Psychological level: a tone is our experience of sound, "*Tonempfindung*"
- Intellectual level: a tone is a position in a tonal system, described in a syntactical meta-language, "*Tonvorstellung*". Mathematical music theory belongs to this realm.

Work of Mazzola and Collaborators

- Mazzola, Guerino. *Gruppen und Kategorien in der Musik. Entwurf einer mathematischen Musiktheorie*. Research and Exposition in Mathematics, 10. Heldermann Verlag, Berlin, 1985.
- Mazzola, Guerino. *The topos of music. Geometric logic of concepts, theory, and performance*. In collaboration with Stefan Göller and Stefan Müller. Birkhäuser Verlag, Basel, 2002.
- Noll, Thomas, *Morphologische Grundlagen der abendländischen Harmonik* in: Moisei Boroda (ed.), *Musikometrika* 7, Bochum: Brockmeyer, 1997.

These developed a mathematical meta-language for music theory, investigated concrete music-theoretical questions, analyzed works of music, and did compositional experiments (Noll 2005).

Lewin's Transformational Theory

- Lewin, David. *Generalized Musical Intervals and Transformations*, Yale University Press, 1987.
- Lewin, David. *Musical Form and Transformation: 4 Analytic Essays*, Yale University Press, 1993.

Transformational analysis asks: which transformations are idiomatic for a given work of music?

Lewin introduced *generalized interval systems* to analyze works of music. The operations of Hugo Riemann were a point of departure. Mathematically, a generalized interval system is a *simply transitive group action.*

Musical Transformations

Examples of musical transformations:

- Transposition
- Inversion
- Retrograde
- Enchaining
- Rhythmic shifts
- Chord inversion: root, first inversion, second inversion

II. Generalized Interval Systems and Simply Transitive Group Actions

Generalized Interval Systems

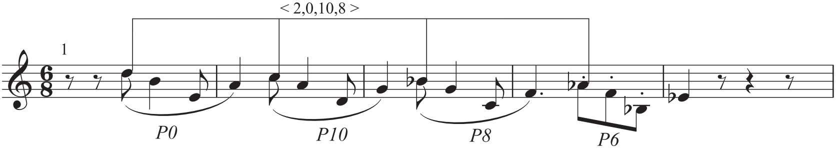
Example

Consider: Set $\mathbb{Z}_{12}$, *Interval Group* $(\mathbb{Z}_{12}, +)$,
Interval Function $int\colon \mathbb{Z}_{12} \times \mathbb{Z}_{12} \to (\mathbb{Z}_{12}, +)$ *(into group)*,
$int(s, t) := t - s$.

1. *Additivity:* $int(2, \mathbf{5}) + int(\mathbf{5}, 7) = int(2, 7)$.
   Proof:
   $int(2, \mathbf{5}) + int(\mathbf{5}, 7) = (\mathbf{5} - 2) + (7 - \mathbf{5}) = 7 - 2 = int(2, 7)$
2. *For given pitch class 2, and given interval 3, there exists a unique pitch class above 2 by interval 3, that is* $int(2, t) = 3$.
   Proof of existence: $t = 5$ *fits the bill.*
   Proof of uniqueness: if $int(2, t) = int(2, t')$, *then* $t - 2 = t' - 2$ *so that* $t = t'$.

Example: Subject of Hindemith, *Ludus Tonalis*, Fugue in *E*

Subject:



$$\langle 2, 11, 4, 9\rangle \ \langle 0, 9, 2, 7\rangle \ \langle 10, 7, 0, 5\rangle \ \langle 8, 5, 10, 3\rangle$$

Same interval content in each cell: -3, -7, +5, with respect to musical space $\mathbb{Z}_{12}$, interval group $(\mathbb{Z}_{12}, +)$, and interval function $int(s, t) := t - s$.

Generalized Interval Systems

Definition (Lewin)

A generalized interval system consists of a set S of musical elements, a mathematical group $IVLS$ which consists of the intervals of the generalized interval system, and an interval function $int : S \times S \to IVLS$ such that:

1. *For all $r, s, t \in S$, we have $int(r,s)int(s,t) = int(r,t)$,*
2. *For every $s \in S$ and every $i \in IVLS$, there exists a unique $t \in S$ such that $int(s,t) = i$.*

Example

$(\mathbb{Z}_{12}, (\mathbb{Z}_{12}, +), int)$ *where* $int(s,t) := t - s$ *as above.*

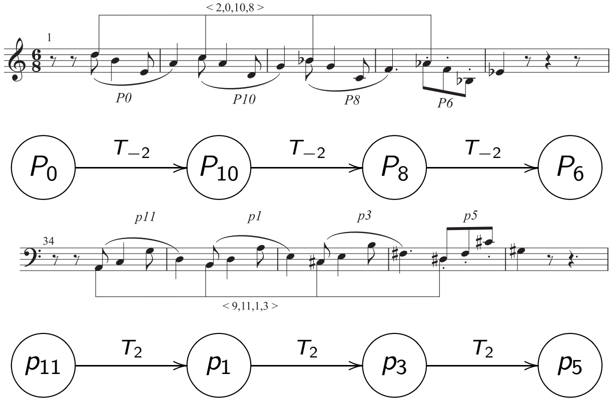
Generalized Interval Systems: Why?

"The point of invoking any generalized interval system is that its consistency serves as a warrant that something coherent can be heard and/or said in the intervallic or transformational terms that the generalized interval system presumes, assuming that the generalized interval system is well-grounded in musical reality."

Quote from page 327 of David Clampitt. "Alternative Interpretations of Some Measures from *Parsifal*." *Journal of Music Theory* 42/2 (1998):321-334.
Note: for transformations of GIS see next slide, for illustration see upcoming Grail example

Transpositions in $(\mathbb{Z}_{12}, (\mathbb{Z}_{12}, +), int)$, Hindemith, Fugue in E

The function $T_i\colon \mathbb{Z}_{12} \to \mathbb{Z}_{12}$ defined by $T_i(s) = s + i$ is called *transposition by i*. Example: See Fiore-Satyendra, *MTO*, 2005.



Generalized Interval Systems=Simply Transitive Group Actions

Example

Consider $(\mathbb{Z}_{12}, (\mathbb{Z}_{12}, +), int)$.
$s + i$ *is the unique element satisfying* $int(s, s + i) = i$.
Thus, transposition T_i *is uniquely defined by* $int(s, T_i(s)) = i$.
$\{T_i\}_{i \in \mathbb{Z}_{12}}$ *acts simply transitively on the musical set* $\mathbb{Z}_{12}$.

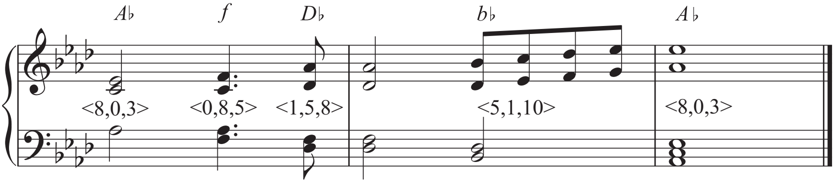
Definition (Lewin)

Suppose $(S, IVLS, int)$ *is a GIS.* $T_i \colon S \to S$ *is transposition by the interval* $i \in IVLS$*, defined by* $int(s, T_i(s)) = i$.

The transpositions form a group called *SIMP* which acts simply transitively on the musical space S.
GIS's are the same thing as simply transitive group actions

A non-Abelian Example from Wagner's *Parsifal* (See Cohn and Clampitt)

"Grail" Theme



A non-Abelian Example from Wagner's *Parsifal* (See Cohn and Clampitt)

S is the *set of chords* $\{G, g, E\flat, e\flat, B, b\}$.
$SIMP \cong \mathrm{Sym}(3)$ is the following group in cycle notation.

$$\begin{array}{ll} \mathrm{Id} = () & LP = (E\flat\ G\ B)(e\flat\ b\ g) \\ P = (E\flat\ e\flat)(G\ g)(B\ b) & PL = (E\flat\ B\ G)(e\flat\ g\ b) \\ L = (E\flat\ g)(G\ b)(B\ e\flat) & PLP = (E\flat\ b)(G\ e\flat)(B\ g) \end{array}$$

Action is simply transitive by inspection.
Maximally smooth cycle:

$$G \stackrel{P}{\longmapsto} g \stackrel{L}{\longmapsto} E\flat \stackrel{P}{\longmapsto} e\flat \stackrel{L}{\longmapsto} B \stackrel{P}{\longmapsto} b \stackrel{L}{\longmapsto} G$$

Perceptual basis of GIS: how the triadic constituents move.

Grail theme: $E\flat \stackrel{PLP}{\longleftrightarrow} b \stackrel{L}{\longleftrightarrow} G \stackrel{PLP}{\longleftrightarrow} e\flat$ (conj=modulation), with $E\flat \stackrel{P}{\longleftrightarrow} e\flat$

III. Duality in the Sense of Lewin

COMM-SIMP Duality

Definition (Lewin)

Suppose $(S, IVLS, int)$ is a generalized interval system. A function $f : S \to S$ is called interval preserving if $int(f(s), f(t)) = int(s, t)$ for all $s, t \in S$. We denote the group of bijective interval preserving functions by COMM.

Example

For $(\mathbb{Z}_{12}, (\mathbb{Z}_{12}, +), int)$, the interval preserving functions are the transpositions. (typical for Abelian case)

COMM-SIMP Duality

Theorem (Lewin)

1. *Let $(S, IVLS, int)$ be a generalized interval system. Then COMM is the group of bijective functions $S \to S$ which commute with the transposition group SIMP. In symbols, $C(SIMP) = COMM$.*
2. *The group COMM acts simply transitively on S, and hence determines a generalized interval system called the dual to $(S, IVLS, int)$. The transposition group of the dual is COMM.*
3. *The group SIMP is the group of interval preserving bijections for the dual, and hence $C(COMM) = SIMP$.*

COMM and SIMP in *Parsifal* Example

S is the *set of chords* $\{G, g, E\flat, e\flat, B, b\}$.
$COMM \cong \text{Sym}(3)$ is the following group in cycle notation.

$$\begin{array}{ll} T_0 = () & I_1 = (E\flat\ e\flat)(G\ b)(B\ g) \\ T_4 = (E\flat\ G\ B)(e\flat\ g\ b) & I_5 = (B\ b)(G\ e\flat)(E\flat\ g) \\ T_8 = (E\flat\ B\ G)(e\flat\ b\ g) & I_9 = (G\ g)(E\flat\ b)(B\ e\flat) \end{array}$$

Action is simply transitive by inspection.
Maximally smooth cycle:

$$G \stackrel{I_9}{\longmapsto} g \stackrel{I_5}{\longmapsto} E\flat \stackrel{I_1}{\longmapsto} e\flat \stackrel{I_9}{\longmapsto} B \stackrel{I_5}{\longmapsto} b \stackrel{I_1}{\longmapsto} G$$

Perceptual basis of GIS: which pitch classes are exchanged.

Grail theme: $E\flat \stackrel{I_9}{\longleftrightarrow} b \stackrel{I_5}{\longleftrightarrow} G \stackrel{I_1}{\longleftrightarrow} e\flat$, with $E\flat \stackrel{I_1}{\longleftrightarrow} e\flat$

IV. Summary

Summary

- Mathematical music theorists use mathematics to analyze works of music, study musical objects, and compose...
- Transformational analysis asks: which transformations are idiomatic for a work of music?
- Generalized interval systems are one of the main tools in transformational theory.
- Generalized interval systems = Simply transitive group actions = torsors
- COMM-SIMP Duality is same as simply transitive groups actions which centralize each other