Classroom Presentation I: Dihedral Groups, Centralizers, and Beethoven on the Torus

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January 9, 2011

Overview

We describe two musical actions of the dihedral group of order 24 on the major and minor triads.

1. Through transpositions and inversions.
2. Through the neo-Riemannian PLR-group.

These two actions are *dual*.

Our Focus

Mathematical tools in neo-Riemannian Music Theory:

- The $\mathbb{Z}_{12}$ Model of Pitch Class
- Transposition and Inversion
- The Dihedral Group
- Centralizers
- The neo-Riemannian *PLR*-group
- Its associated graphs.

Reference

This talk summarizes:

1. Alissa Crans, Thomas M. Fiore, and Ramon Satyendra. Musical actions of dihedral groups. *American Mathematical Monthly*, Volume 116, Number 6, June-July 2009, pp. 479-495.

Available at
`http://www-personal.umd.umich.edu/~tmfiore/`
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I. Translating Music to Algebra

The $\mathbb{Z}_{12}$ Model of Pitch Class

We have a bijection between the set of pitch classes and $\mathbb{Z}_{12}$.

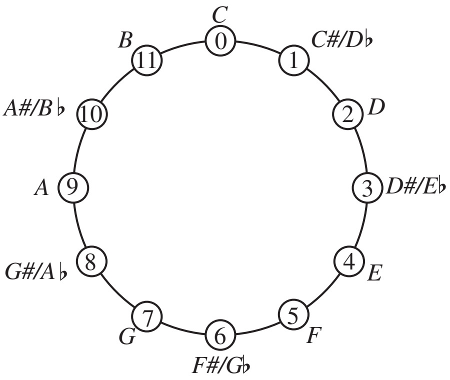


Figure: The musical clock.

Transposition

The bijective function

$$T_n : \mathbb{Z}_{12} \to \mathbb{Z}_{12}$$

$$T_n(x) := x + n$$

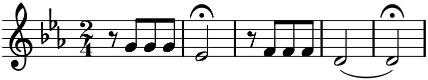
is called *transposition* by musicians, or *translation* by mathematicians.

$$T_{-4}(7) = 7 - 4 = 3$$

$$T_{-3}(5) = 5 - 3 = 2$$

G	G	G	$E\flat$
7	7	7	3
7	7	7	$T_{-4}(7)$

F	F	F	D
5	5	5	2
5	5	5	$T_{-3}(5)$



Inversion

The bijective function

$$I_n : \mathbb{Z}_{12} \to \mathbb{Z}_{12}$$

$$I_n(x) := -x + n$$

is called *inversion* by musicians, or *reflection* by mathematicians.

$$I_0(0) = -0 = 0$$

$$I_0(7) = -7 = 5$$

$\langle C, G \rangle$	$\langle C, F \rangle$
$\langle 0, 7 \rangle$	$\langle I_0(0), I_0(7) \rangle$
$\langle 0, 7 \rangle$	$\langle 0, 5 \rangle$

The T/I-Group

Altogether, these transpositions and inversions form the *T/I-group*.

This is the group of symmetries of the 12-gon, the *dihedral group* of order 24.

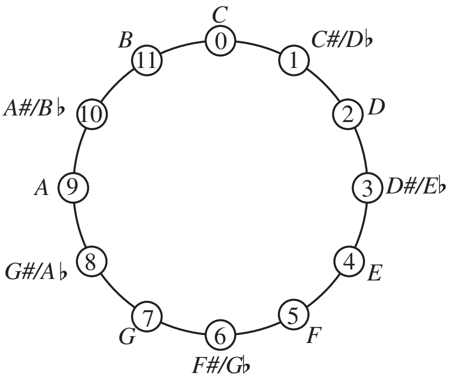


Figure: The musical clock.

Major and Minor Triads

Major and minor triads are very common in Western music.

$$C\text{-major} = \langle C, E, G \rangle = \langle 0, 4, 7 \rangle$$

$$c\text{-minor} = \langle G, E\flat, C \rangle = \langle 7, 3, 0 \rangle$$

The set S of consonant triads	
Major Triads	Minor Triads
$C = \langle 0,4,7 \rangle$	$\langle 0,8,5 \rangle = f$
$C\sharp = D\flat = \langle 1,5,8 \rangle$	$\langle 1,9,6 \rangle = f\sharp = g\flat$
$D = \langle 2,6,9 \rangle$	$\langle 2,10,7 \rangle = g$
$D\sharp = E\flat = \langle 3,7,10 \rangle$	$\langle 3,11,8 \rangle = g\sharp = a\flat$
$E = \langle 4,8,11 \rangle$	$\langle 4,0,9 \rangle = a$
$F = \langle 5,9,0 \rangle$	$\langle 5,1,10 \rangle = a\sharp = b\flat$
$F\sharp = G\flat = \langle 6,10,1 \rangle$	$\langle 6,2,11 \rangle = b$
$G = \langle 7,11,2 \rangle$	$\langle 7,3,0 \rangle = c$
$G\sharp = A\flat = \langle 8,0,3 \rangle$	$\langle 8,4,1 \rangle = c\sharp = d\flat$
$A = \langle 9,1,4 \rangle$	$\langle 9,5,2 \rangle = d$
$A\sharp = B\flat = \langle 10,2,5 \rangle$	$\langle 10,6,3 \rangle = d\sharp = e\flat$
$B = \langle 11,3,6 \rangle$	$\langle 11,7,4 \rangle = e$

Major and Minor Triads

The T/I-group acts on the set S of major and minor triads componentwise.

$$T_1\langle 0,4,7\rangle = \langle T_1 0, T_1 4, T_1 7\rangle = \langle 1,5,8\rangle$$

$$I_0\langle 0,4,7\rangle = \langle I_0 0, I_0 4, I_0 7\rangle = \langle 0,8,5\rangle$$

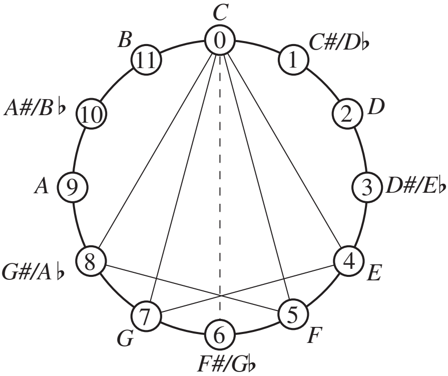


Figure: I_0 applied to a C-major triad yields an f-minor triad.

II. Some Group Theory

The Dihedral Group of Order 24

The *dihedral group of order 24* is the group of symmetries of a regular 12-gon.

Algebraically, the *dihedral group of order 24* is the group generated by two elements, s and t, subject to the three relations

$$s^{12} = 1, \qquad t^2 = 1, \qquad tst = s^{-1}.$$

The T/I-group is (isomorphic to) the dihedral group of order 24.

Group Actions

An *action of a group* G on a set S is a function

$$G \times S \longrightarrow S$$

$$(g, s) \longmapsto gs$$

such that $g(hs) = (gh)s$ and $es = s$ for all $g, h \in G$ and all $s \in S$. This is the same as a group homomorphism $G \longrightarrow \mathrm{Sym}(S)$, where $\mathrm{Sym}(S)$ is the *symmetric group* on S.

Example

$S =$ *the set of major and minor triads*
$G = T/I$*-group*
$T_n(\langle x, y, z\rangle) = \langle T_n x, T_n y, T_n z\rangle \qquad I_n(\langle x, y, z\rangle) = \langle I_n x, I_n y, I_n z\rangle$

The Orbit-Stabilizer Theorem

The *orbit* of an element Y of a set S under a group action of G on S consists of all those elements of S to which Y is moved, in other words

$$\text{orbit of } Y = \{hY \mid h \in G\}.$$

The *stabilizer group* of Y consists of all those elements of G which fix Y, namely

$$G_Y = \{h \in G \mid hY = Y\}.$$

Theorem (Orbit-Stabilizer Theorem)

If a finite group G acts on a set S and $Y \in S$, then

$$|G|/|G_Y| = |\text{orbit of } Y|.$$

The Orbit-Stabilizer Theorem

Example

S = the set of major and minor triads
G = T/I-group, acting componentwise, and transitively

$$24/|G_Y| = |G|/|G_Y| = |\text{orbit of } Y| = 24$$

Thus $|G_Y| = 1$ for each Y in S,
and the-T/I group acts simply transitively:
for each Y and Z in S, there exists a unique $g \in G$ such that $gY = Z$.

Duality in the Sense of Lewin

The *centralizer* of a subgroup G of $\mathrm{Sym}(S)$ is the set of elements of $\mathrm{Sym}(S)$ which commute with all elements of G, namely

$$C(G) = \{\sigma \in \mathrm{Sym}(S) \mid \sigma g = g\sigma \text{ for all } g \in G\}.$$

Subgroups G_1 and G_2 of $\mathrm{Sym}(S)$ are *dual* if each acts simply transitively on S and

$$C(G_1) = G_2 \quad \text{and} \quad C(G_2) = G_1.$$

Duality in the Sense of Lewin

Example

Let G be a group and consider the left and right Cayley embeddings $\lambda, \rho\colon G \to Sym(G)$.

$$\lambda_g(s) := gs$$
$$\rho_h(s) := sh^{-1}$$

Then $\lambda(G)$ and $\rho(G)$ act simply transitively and

$$\lambda_g \rho_h(s) = gsh^{-1} = \rho_h \lambda_g(s).$$

Further, $\lambda(G)$ and $\rho(G)$ are dual groups.
In fact, all dual groups are of this form.

III. The neo-Riemannian Group

Neo-Riemannian Music Theory

- Recent work focuses on the neo-Riemannian operations P, L, and R.
- P, L, and R generate a dihedral group, called the *neo-Riemannian group*. As we'll see, this group is *dual* to the T/I-group in the sense of Lewin.
- These transformations arose in the work of the 19th century music theorist Hugo Riemann, and have a pictorial description on the *Oettingen/Riemann Tonnetz*.
- P, L, and R are defined in terms of common tone preservation.

The neo-Riemannian Transformation P

We consider three functions

$$P, L, R : S \to S.$$

Let $P(x)$ be that triad of opposite type as x with the first and third notes switched.

For example

$P\langle \mathbf{0}, 4, \mathbf{7} \rangle =$

$P(C\text{-major}) =$

The set S of consonant triads	
Major Triads	Minor Triads
$C = \langle 0, 4, 7 \rangle$	$\langle 0, 8, 5 \rangle = f$
$C\sharp = D\flat = \langle 1, 5, 8 \rangle$	$\langle 1, 9, 6 \rangle = f\sharp = g\flat$
$D = \langle 2, 6, 9 \rangle$	$\langle 2, 10, 7 \rangle = g$
$D\sharp = E\flat = \langle 3, 7, 10 \rangle$	$\langle 3, 11, 8 \rangle = g\sharp = a\flat$
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$A\sharp = B\flat = \langle 10, 2, 5 \rangle$	$\langle 10, 6, 3 \rangle = d\sharp = e\flat$
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The neo-Riemannian Transformation P

We consider three functions

$$P, L, R : S \to S.$$

Let $P(x)$ be that triad of opposite type as x with the first and third notes switched.

For example

$$P\langle \mathbf{0}, 4, \mathbf{7} \rangle = \langle \mathbf{7}, 3, \mathbf{0} \rangle$$

$$P(C\text{-major}) = c\text{-minor}$$

The set S of consonant triads	
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$A\sharp = B\flat = \langle 10, 2, 5 \rangle$	$\langle 10, 6, 3 \rangle = d\sharp = e\flat$
$B = \langle 11, 3, 6 \rangle$	$\langle 11, 7, 4 \rangle = e$

The neo-Riemannian Transformations L and R

- Let $L(x)$ be that triad of opposite type as x with the second and third notes switched. For example

$$L\langle 0, \mathbf{4}, \mathbf{7} \rangle = \langle 11, \mathbf{7}, \mathbf{4} \rangle$$

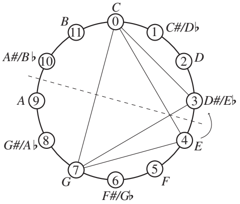
$$L(C\text{-major}) = e\text{-minor}.$$

- Let $R(x)$ be that triad of opposite type as x with the first and second notes switched. For example

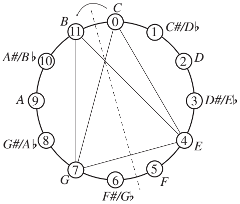
$$R\langle \mathbf{0}, \mathbf{4}, 7 \rangle = \langle \mathbf{4}, \mathbf{0}, 9 \rangle$$

$$R(C\text{-major}) = a\text{-minor}.$$

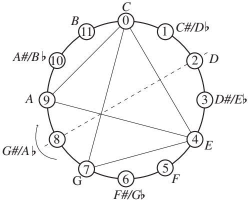
Minimal motion of the moving voice under P, L, and R.



$PC = c$

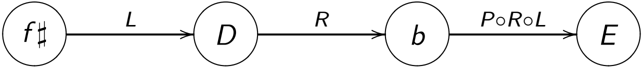


$LC = e$



$RC = a$

Example: “Oh! Darling” from the Beatles



E+ *A* *E*
Oh____ Darling please believe me
f♯ *D*
I’ll never do you no harm
*b*7 *E*7
Be-lieve me when I tell you
*b*7 *E*7 *A*
I’ll never do you no harm

The neo-Riemannian PLR-Group

Definition

The neo-Riemannian PLR-group is the subgroup of $Sym(S)$ *generated by* P, L, *and* R.

Various relations:

$$P^2 = L^2 = R^2 = 1$$

$$R(LR)^3 = P$$

$$(LR)^{12} = 1$$

$$\ldots$$

The Structure of the neo-Riemannian *PLR*-Group

Theorem (Lewin 80's, Hook 2002, ...)

The PLR-group is dihedral of order 24 and is generated by L and R.

Proof.(CFS) P,L, and R commute with T_1.
Apply R and L to C-major to get: $C, a, F, d, B\flat, g, E\flat, c, A\flat, f, D\flat, b\flat, G\flat, e\flat, B, g\sharp, E, c\sharp, A, f\sharp, D, b, G, e, C$. Thus, the 24 bijections $R, LR, RLR, \ldots, R(LR)^{11}$, and $(LR)^{12} = 1$ are distinct, the PLR-group has at least 24 elements, and that LR has order 12.
Further $R(LR)^3(C) = c$, and since $R(LR)^3$ has order 2 and commutes with T_1, we see that $R(LR)^3 = P$, and the PLR-group is generated by L and R alone.
If we set $s = LR$ and $t = L$, then $s^{12} = 1, t^2 = 1$, and $tst = s^{-1}$.
Finally, the PLR-group is dihedral of order 24. QED

The neo-Riemannian *PLR*-Group and Duality

Corollary

The PLR-group acts simply transitively on the set of consonant triads.

Theorem (Lewin 80's, Hook 2002, ...)

The PLR-group is dual to the T/I-group.

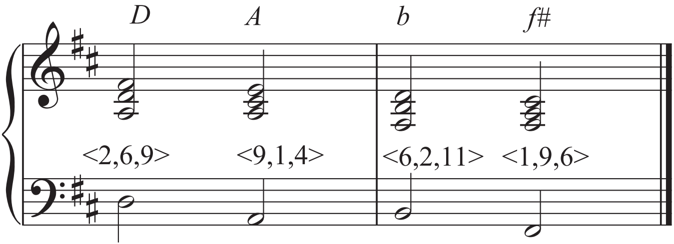
Proof.(CFS) P,L, and R commute with T_1 and I_0, so $PLR \leq C(T/I)$.

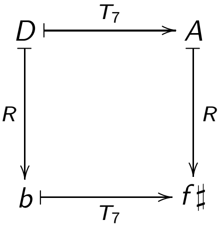
$$|C(T/I)|/|C(T/I)_Y| = |\text{orbit of } Y| \leq |S| = 24$$

$$24 \leq |PLR| \leq |C(T/I)| \leq 24$$

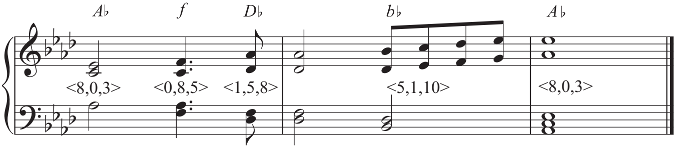
since $|C(T/I)_Y| = 1$. Thus, $PLR = C(T/I)$.
Similarly, $T/I = C(PLR)$. QED

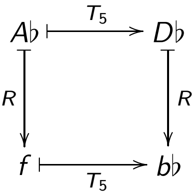
Example of Duality: Pachelbel's *Canon in D*, ca 1680





Example of Duality: Wagner's *Parsifal*, 1882

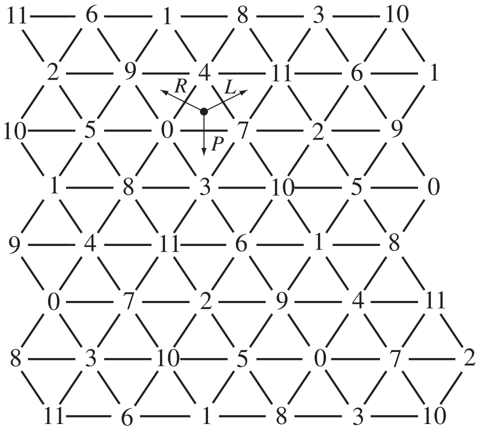




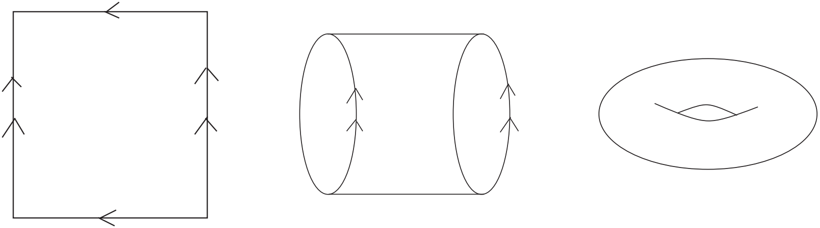
(Here we consider the duality between the 24-element groups, rather than the duality between the 6-element groups in the Overview Presentation.)

IV. The Geometry of the neo-Riemannian Group

The Oettingen/Riemann *Tonnetz*



The Torus



The Dual Graph to the Tonnetz

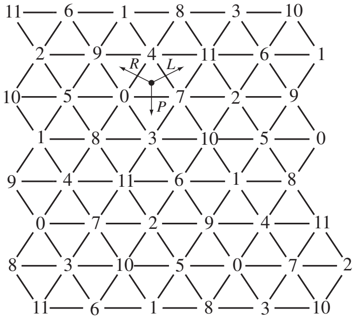


Figure: The *Tonnetz*.

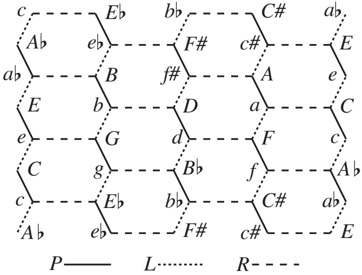


Figure: Douthett and Steinbach's Graph.

The Dual Graph to the Tonnetz

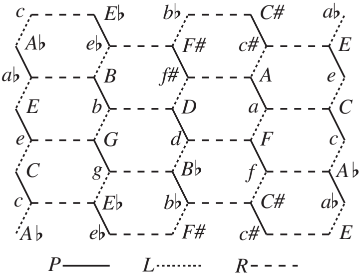


Figure: Douthett and Steinbach's Graph.

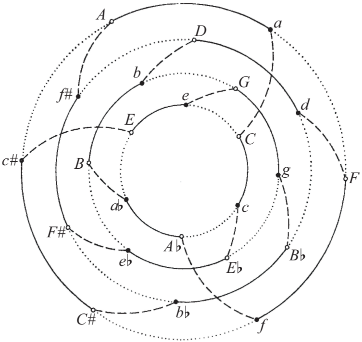
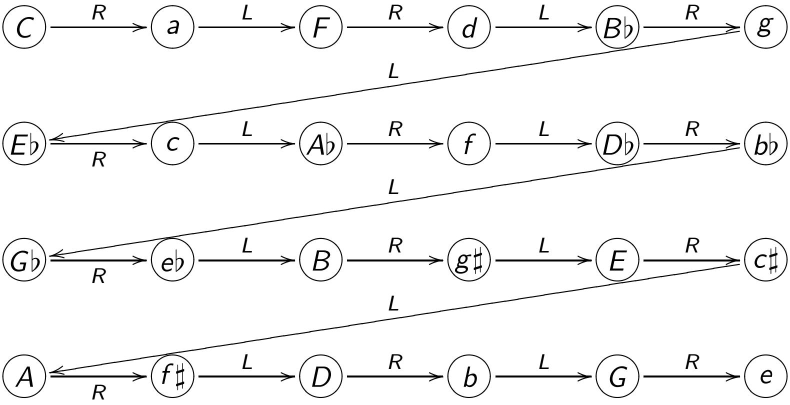


Figure: Waller's Torus.

Beethoven's 9th, 2nd Mvmt, Measures 143-17 (Cohn)



We have seen:

- How to encode pitch classes as integers modulo 12, and consonant triads as 3-tuples of integers modulo 12
- How the T/I-group acts componentwise on consonant triads
- How the PLR-group acts on consonant triads
- Duality between the T/I-group and the PLR-group
- Geometric depictions on the torus and musical examples.

But most music does not consist entirely of triads! See the extension of Fiore-Satyendra in Music Theory Online, Volume 11, Number 3, September 2005.
Also, not every musical action is simply transitive. See the extension of R. Peck in the Journal of Music Theory, next talk.