ALGEBRA PROBLEMS FOR MAA MINI-COURSE ON GEOMETRY AND ALGEBRA IN MUSIC THEORY JOINT MATHEMATICS MEETING IN NEW ORLEANS, JANUARY 9, 2011

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1. Pitches and Pitch Classes

(1.1) (**Pitch Classes and Frequency**) If a and b are pitches designated by their frequency, we write

$$a \sim b$$

if $a/b = 2^j$ for some $j \in \mathbb{Z}$, in other words if a and b are a whole number of *octaves* apart. Prove that this defines an *equivalence relation*, in other words show
$a \sim a$ for all pitches a
$a \sim b$ implies $b \sim a$ for all pitches a and b
$a \sim b$ and $b \sim c$ implies $a \sim c$ for all pitches a, b, and c.
The equivalences classes for this equivalence relation are called *pitch classes*.

(1.2) (**Chromatic Pitches and Chromatic Pitch Classes**) We may think of the set $\mathbb{Z}$ as the set of chromatic, equal tempered *pitches*, with 0 being middle C. The number 12 is the C above middle C. The pitches of the keys of a piano correspond to a subset of $\mathbb{Z}$. We write

$$a \sim b$$

if $b - a$ is divisible by 12. Prove that this defines an equivalence relation. How is it related to the equivalence relation in 1.1?
The equivalences classes for this equivalence relation are also called *pitch classes*. Thus $\mathbb{Z}_{12}$ is called the collection pitch classes (though of course there are many more pitch classes as we saw in 1.1.)

2. Transposition and Inversion

Recall the *transposition* and *inversion* functions.

$$T_n : \mathbb{Z}_{12} \longrightarrow \mathbb{Z}_{12}\ ,\ \ T_n(x) := x + n \bmod 12$$

$$I_n : \mathbb{Z}_{12} \longrightarrow \mathbb{Z}_{12}\ ,\ \ I_n(x) := -x + n \bmod 12.$$

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(2.1) (**Circle of Fifths, Circle of Fourths**) Apply T_7 repeatedly to 0 (remember to use arithmetic modulo 12). This is the *circle of fifths*. Why should you expect to get 12 pitch classes? Apply T_5 repeatedly to 0. This is the *circle of fourths*. Why again should you expect to get 12 pitch classes?

(2.2) (**Relations in the T/I-Group**) Verify the relations

$$T_m \circ T_n = T_{m+n \text{ mod } 12}$$
$$T_m \circ I_n = I_{m+n \text{ mod } 12}$$
$$I_m \circ T_n = I_{m-n \text{ mod } 12}$$
$$I_m \circ I_n = T_{m-n \text{ mod } 12}$$

and conclude that the set of functions $\{T_n | n \in \mathbb{Z}_{12}\} \cup \{I_n | n \in \mathbb{Z}_{12}\}$ forms a group. This group is called the *T/I-group* in the neo-Riemannian literature.

(2.3) (**Group Structure of the T/I-Group**) Recall the geometric proof that the T/I-group is (isomorphic to) the dihedral group of order 24, that is, the transpositions and inversions are the symmetries of the regular 12-gon. Prove algebraically that the T/I-group is (isomorphic to) the dihedral group of order 24 by verifying the generators and relations presentation of the dihedral group, that is, find s and t in the T/I-group such that

$$s^{12} = 1, \qquad t^2 = 1, \qquad tst = s^{-1}.$$

(2.4) (**Set Classes**) Prove that the T/I-group acts on the set of subsets of $\mathbb{Z}_{12}$. The orbits under this action are called *set classes*. Note that an orbit is a set of sets, and an element of an element of an orbit is a pitch class. The next problem should clarify this point.

(2.5) (**Example of a Set Class**) Write down the set class of $\{0, 4, 7\}$. Can you say what this orbit is in musical terms? (Hint: use the dictionary to translate the numbers back into pitch classes).

3. Group Theory

(3.1) (**Group Actions and Orbits**) If G is a group and X is a set, then a *(left) group action of G on X* is a function

$$G \times X \to X$$

such that

$$g_1(g_2x) = (g_1g_2)x$$
$$1_Gx = x$$

for all $g_1, g_2 \in G$ and $x \in X$. For $x, y \in X$, we write $x \sim y$ if there exists some $g \in G$ such that $gx = y$. Prove that this is an equivalence relation, in other words
$x \sim x$
$x \sim y$ implies $y \sim x$

$x \sim y$ and $y \sim z$ implies $x \sim z$
for all $x, y, z \in X$. An *equivalence class*

$$[x] = \{y \in X | y \sim x\}$$

is called an *orbit*.

(3.2) (**Group Actions as Homomorphisms into Symmetric Groups**) Prove that an action of a group G on a set X is the same as a group homomorphism $\rho\colon G \to \mathrm{Sym}(X)$. Here $\mathrm{Sym}(X)$ denotes the the *symmetric group on the set* X, that is, the group of bijections $X \to X$ with group operation given by composition. Hint: to define ρ, assign to $g \in G$ the bijection $x \mapsto gx$.

(3.3) (**Simple Transitivity and Left/Right Multiplication**) A group action is *transitive* if for any $x, y \in X$ there exists a $g \in G$ such that $gx = y$, in other words there is only one orbit and that orbit is all of X. A group action is *simple* or *free* if

$$gx = g'x \text{ for some } x \in X \Rightarrow g = g'.$$

A group action is *simply transitive* if it is both transitive and simple. More succinctly, a group action is simply transitive if and only if for each x and each y there exists a unique $g \in G$ such that $gx = y$. Show that any group acts simply transitively on itself via left multiplication. Similarly, show any group acts simply transitively on itself (from the left) via right multiplication by the inverse, that is $g \cdot x := xg^{-1}$.

(3.4) (**Example of Simple Transitivity**) Let S be the following **set of chords**.

$$\{G, g, E\flat, e\flat, B, b\}$$

Let *SIMP* $\cong \mathrm{Sym}(3)$ be the following group in cycle notation.

$$\begin{array}{ll} \mathrm{Id} = () & LP = (E\flat\ G\ B)(e\flat\ b\ g) \\ P = (E\flat\ e\flat)(G\ g)(B\ b) & PL = (E\flat\ B\ G)(e\flat\ g\ b) \\ L = (E\flat\ g)(G\ b)(B\ e\flat) & PLP = (E\flat\ b)(G\ e\flat)(B\ g) \end{array}$$

Prove that *SIMP* acts simply transitively on S. There are several ways to do this, some more efficient than others... The fastest way is to probably to use the Orbit-Stabilizer Theorem as in the next problem. The set of pitch classes underlying the set of chords $\{G, g, E\flat, e\flat, B, b\}$ is called a *hexatonic system*. There are in four hexatonic systems, they correspond to the four orbits of the PL-group acting on the set of major and minor chords.

(3.5) (**Orbit-Stabilizer Theorem and Simple Transitivity**) If a group G acts on a set X, then the *stabilizer of* $x \in X$ is the group

$$G_x := \{g \in G | gx = x\}.$$

The Orbit-Stabilizer Theorem says the following.

Theorem 3.1. *If a finite group G acts on a set X, then*

$$|G|/|G_x| = |\text{orbit of } x|$$

where $|\cdot|$ means "number of elements."

The finiteness assumption can be left off by changing the statement into a bijection between cosets and equivalence classes. Use the Orbit-Stabilizer Theorem to prove the following.

(a) If a finite group G acts transitively on a finite set X and $|X| = |G|$, then G acts simply transitively.

(b) If a finite group G acts simply transitively on a set X, then $|X| = |G|$.

(3.6) (**Dual Groups and the Left/Right Regular Representations**) Let $\mathrm{Sym}(S)$ be the symmetric group on the set S. Two subgroups G and H of the symmetric group $\mathrm{Sym}(S)$ are called *dual* if their natural actions on S are simply transitive and each is the centralizer of the other, that is

$$C_{\mathrm{Sym}(S)}(G) = H \quad \text{and} \quad C_{\mathrm{Sym}(S)}(H) = G.$$

Consider now $S = G$ and the left and right regular representations $\lambda, \rho \colon G \to \mathrm{Sym}(G)$, which are defined by $\lambda_g(h) = gh$ and $\rho_g(h) = hg^{-1}$. Recall that λ and ρ are embeddings. We denote the images by $\lambda(G)$ and $\rho(G)$. In problem 3.3 you proved that the natural actions of the groups $\lambda(G)$ and $\rho(G)$ are simply transitive. Prove in one line that the groups $\lambda(G)$ and $\rho(G)$ commute. Next, prove that they actually centralize one-another. The pair $\lambda(G)$ and $\rho(G)$ are the fundamental example of dual groups.

4. The Neo-Riemannian Group

Recall the neo-Riemannian operations P, L, and R. The triad $P(x)$ is that triad of opposite type as x with the first and third notes switched. The triad $L(x)$ is that triad of opposite type as x with the second and third notes switched. The triad $R(x)$ is that triad of opposite type as x with the first and second notes switched.

(4.1) (**Computation of P, L, and R**) Calculate $P\langle 1,5,8\rangle$, $L\langle 10,6,3\rangle$, and $R\langle 9,1,4\rangle$ using the table of major and minor triads below.

Major Triads	Minor Triads
$C = \langle 0,4,7\rangle$	$\langle 0,8,5\rangle = f$
$C\sharp = D\flat = \langle 1,5,8\rangle$	$\langle 1,9,6\rangle = f\sharp = g\flat$
$D = \langle 2,6,9\rangle$	$\langle 2,10,7\rangle = g$
$D\sharp = E\flat = \langle 3,7,10\rangle$	$\langle 3,11,8\rangle = g\sharp = a\flat$
$E = \langle 4,8,11\rangle$	$\langle 4,0,9\rangle = a$
$F = \langle 5,9,0\rangle$	$\langle 5,1,10\rangle = a\sharp = b\flat$
$F\sharp = G\flat = \langle 6,10,1\rangle$	$\langle 6,2,11\rangle = b$
$G = \langle 7,11,2\rangle$	$\langle 7,3,0\rangle = c$
$G\sharp = A\flat = \langle 8,0,3\rangle$	$\langle 8,4,1\rangle = c\sharp = d\flat$
$A = \langle 9,1,4\rangle$	$\langle 9,5,2\rangle = d$
$A\sharp = B\flat = \langle 10,2,5\rangle$	$\langle 10,6,3\rangle = d\sharp = e\flat$
$B = \langle 11,3,6\rangle$	$\langle 11,7,4\rangle = e$

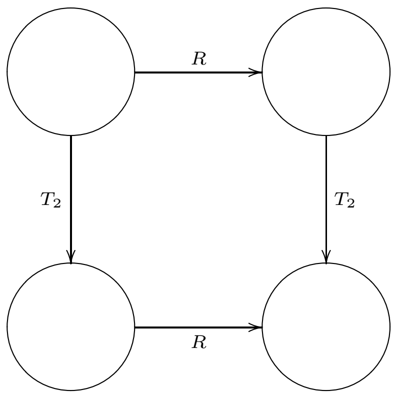
(4.2) (**Order of P, L, and R**) Determine the order of P, L, and R.

(4.3) (**Commutative vs. Noncommutative Diagrams**) Does the diagram

$$\begin{array}{ccc} \mathbb{Z}_{12} & \xrightarrow{I_0} & \mathbb{Z}_{12} \\ {\scriptstyle T_2}\downarrow & & \downarrow{\scriptstyle T_2} \\ \mathbb{Z}_{12} & \xrightarrow[I_0]{} & \mathbb{Z}_{12} \end{array}$$

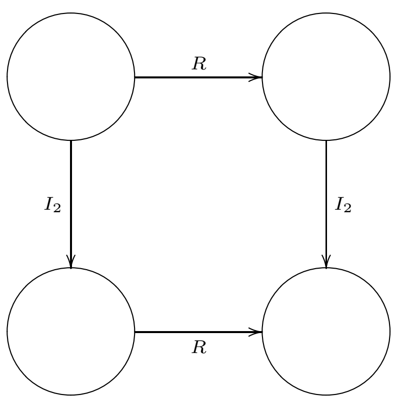
commute? In other words, is the equation $T_2 I_0 = I_0 T_2$ true?

(4.4) (**Dual Groups and Networks**) Insert $\langle 0, 4, 7 \rangle$ into the upper left circle of the network



and verify that the result is the same no matter which path you take. Compare this with Problem 4.3 above. The contextual inversion R is in the dual group to the T/I-group, but the inversion I_0 is not. What else is different about the two examples, e.g., how are the inputs different?

(4.5) (**Dual Groups and Networks**) Insert $\langle 0, 4, 7 \rangle$ into the upper left circle of the network



and verify that the result is the same no matter which path you take. Compare this with Problem 4.3 above.

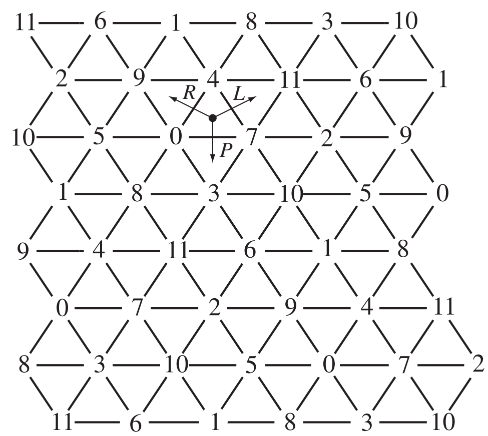


FIGURE 1. The Oettingen/Riemann *Tonnetz* is a tiling of the plane

(4.6) (***LR*-chain in Beethoven**) Calculate each of the following.

$$R\langle 0,4,7\rangle$$
$$L \circ R\langle 0,4,7\rangle$$
$$R \circ L \circ R\langle 0,4,7\rangle$$
$$L \circ R \circ L \circ R\langle 0,4,7\rangle$$

Next translate the results into chord names using the table above. How does this relate to the chord progression in the second movement of Beethoven's Ninth Symphony? Use the chord progression in Beethoven's Ninth Symphony to determine the order of the element LR in the PLR-group.

5. Geometry of the neo-Riemannian Group

(5.1) (**Fundamental Domain for the Tonnetz**)
Recall that the Oettingen/Riemann *Tonnetz* is a tiling of the plane (Riemann did not use equal-tempered tuning, nor the attendant enharmonic equivalence). Figure 1 indicates the *Tonnetz*, which extends infinitely up and down and left and right. However, this is not a true representation of the Oettingen/Riemann *Tonnetz* because vertices are written as pitch classes instead of pitches. Vertices that are identified in the *neo-Riemannian Tonnetz* have the same number in Figure 1. The *neo-Riemannian Tonnetz* is a tiling of the torus, it is the quotient of the Oettingen/Riemann *Tonnetz* by pitch-class equivalence. Problem: draw a parallelogram on Figure 1 which is a *fundamental domain* for the quotient, that is, draw a parallelogram which gives the torus when the top and bottom and also the left and right are identified, but no points in the interior are identified. Recent work of Marek Žabka in the *Journal of Mathematics and Music* investigates the musical ramifications of choosing different fundamental domains.

(5.2) (**Geometry of P, L, R Operations**) Say in words what the P, L, and R operations do to the triangles on the neo-Riemannian *Tonnetz*.

(5.3) (**Word Metric**) Use the fundamental domain you drew in Problem 5.1 to find an upper bound on the minimum number of P, L, R moves it takes to get from one triangle to another.