$\mathbf{Sets}^{\mathcal{G}}$

The Topos of Triads

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Talk by Thomas Fiore

In this talk we find music-theoretical interpretations of topos-theoretical facts. I'll begin with some motivation for why the topos of triads is of interest. Then I'll recall what a topos is with some examples. Then we'll talk about the specifics of the topos of triads, and look at a musical example.

as elementary as possible

like cc don't climax

linear map

Motivation: Why is $\mathbf{Sets}^{\mathcal{G}}$ of interest?

the topos of triads $= \mathbf{Sets}^{\mathcal{G}}$

Consider the monoid $\mathcal{M}$ of affine transformations $\mathbb{Z}_{12} \to \mathbb{Z}_{12}$.

gp w/o inverses

Examples?

Q. How many are there?

$z \xrightarrow{{}^tS} sz + t \qquad (s, t \in \mathbb{Z}_{12} \text{ fixed}) \qquad |\mathcal{M}| = 144$

An affine transformation is determined by what it does to 0, 1.

(Why study affine transformations? Well, the C-major chord can be obtained by iteratively applying a certain affine transformation [in the circle of semitones encoding]).

Encode the pitch classes using the circle of fifths:

C	G	D	A	E	B	F#	C#	G#	Eb	Bb	F
0	1	2	3	4	5	6	7	8	9	10	11

${}^tS(0) = t \qquad t = \text{translation}$

${}^tS(1) = s \cdot 1 + t \qquad s = \text{stretching factor.}$

${}^tS$ maps the fifth $\underset{0,1}{C, G}$ to $t, s+t$.

$\mathcal{M} \leftrightarrow \mathbb{Z}_{12} \times \mathbb{Z}_{12}$

${}^a(b-a) \leftarrow (a, b)$

${}^tS \longmapsto (t, s+t)$

To any pair (a,b) we can associate the affine map which takes (C, G) to (a, b).

So the monoid $\mathcal{M}$ reflects in some sense relations between intervals: For each ordered interval we have an affine transformation and for each affine transformation we have an ordered interval.

One lesson of 20th century mathematics was the utility of Felix Klein's Erlangen program. It says: if you're studying geometry (or anything else) you should consider the maps preserving a substructure. The algebra of those maps reflects the geometry of the substructures. For example, in Galois theory you study field extensions and linear maps which preserve the subfield. Then there is an interesting relationship between intermediate fields and subgroups of the Galois group.

So the algebra of those maps preserving a substructure reflects the geometry of the field containment.

So a reasonable thing to do is to consider the monoid of affine maps preserving the C-major chord.

$\parallel$

triadic monoid

Erlangen Program $\Rightarrow$ study $\mathcal{T} := \{h \in \mathcal{M} \mid h(\{0,1,4\}) \subseteq \{0,1,4\}\}$

$$\begin{array}{cccccccccc} & a & b & c & e & f & f^2 & g & g^2 & \\ & \parallel & \parallel & \parallel & \parallel & \parallel & \parallel & \parallel & \parallel & \\ \mathcal{T} = \{ & {}^{0}0, & {}^{1}0, & {}^{4}0, & {}^{0}1, & {}^{1}3, & {}^{4}9, & {}^{4}8, & {}^{0}4 & \} \end{array}$$

$\underbrace{a,\ b,\ c}_{\text{constant maps}}$, $\underbrace{e}_{\text{identity}}$

f, g generate $\mathcal{T}$.

$(0,0), (1,1), (4,4), (0,1), (1,4), (4,1), (4,0), (0,4)$

$(C,C), (G,G), (E,E), (C,G), (G,E), (E,G), (E,C), (C,E)$

$\parallel$

all consonant pairs in $\{C,E,G\} \times \{C,E,G\}$!!! This is really quite special.

$\mathcal{T}$ acts on $\mathbb{Z}_{12}$ naturally. $\Rightarrow$ $\mathbf{Sets}^{\mathcal{T}}$ is of interest.

$\parallel$

category of sets w/ $\mathcal{T}$-action and equivariant maps

$\parallel$

category of functors $\mathcal{T} \to \mathbf{Sets}$ and natural transformations.

$\mathbf{Sets}^{\mathcal{T}}$ is a <u>topos</u> of presheaves.

Topos Theory

The notion of topos simultaneously generalizes the notions of topological space and universe of sets. This reflects its two different origins in the 1960's: Grothendieck's work in algebraic geometry and topology, as well as Lawvere's foundational investigations.

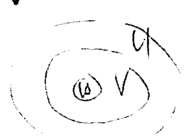
Ex X top. space. $\mathcal{O}(X) :=$ category of open subsets of X.

For example, the real numbers

$\forall U \in \mathcal{O}(X)$, $C(U) := \{ f : U \to \mathbb{R} \mid f \text{ continuous} \}$.

If $V \subseteq U$ in $\mathcal{O}(X)$, then
$$C(U) \longrightarrow C(V)$$
$$f \longmapsto f|_V$$

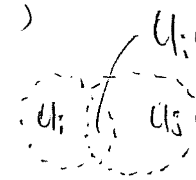
$C : \mathcal{O}(X)^{op} \longrightarrow \mathrm{Sets}$ is a functor.



Additionally, if $U = \bigcup_i U_i$, $U, U_i \in \mathcal{O}(X)$

and $f_i : U_i \xrightarrow{\text{cont.}} \mathbb{R}$ w/ $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$, then

$\exists!\ f : U \longrightarrow \mathbb{R}$ s.t. $f|_{U_i} = f_i$.



Thus C is a <u>sheaf</u>!

Can anyone think of other examples of sheaves?

What about continuous bounded functions

$Sh(X) :=$ category of sheaves on X

$Sh(X)$ is a topos. So from a topological space, we get a topos, and in this sense a topos is a generalized space.

Grothendieck and his collaborators developed a cohomology of such things, which eventually led to a solution of the Weil conjectures.

On the other hand, we have set theory.

Ex The category of Sets has some interesting properties

- all finite limits exist in Sets
- For any inclusion $A \subseteq X$, there is a unique function $X \xrightarrow{\chi_A} \{0,1\}$ such that

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook] & \{0\} \arrow[d] \\ X \arrow[r, "\chi_A"'] & \{0,1\} = \Omega \end{tikzcd}$$

= terminal object

univ. monomorphism

called "subobject classifier"

is a pullback.

(in fact $\chi_A(x) := \begin{cases} 0 & x \in A \\ 1 & x \in X \setminus A \end{cases}$ is the usual characteristic function.

(i.e. there is a bijection $\{A \mid A \subseteq X\} \longleftrightarrow \{\chi : X \to \{0,1\} \mid \chi \text{ func.}\}$

$A \longmapsto \chi_A$

$\{x \in X \mid \chi(x) = 0\} \longleftarrow\!\shortmid\ \chi$)

bijection between ...

- there is a natural bijection

$$\mathrm{Hom}(X \times Y, Z) \cong \mathrm{Hom}(X, \mathrm{Map}(Y, Z))$$

$$\left((x,y) \overset{f}{\longmapsto} f(x,y)\right) \longmapsto \left(x \longmapsto \left(y \overset{f(x,-)}{\longmapsto} f(x,y)\right)\right)$$

These are the three axioms ~~of~~ an elementary topos (or just topos)

Sets is a topos, so the notion of topos generalizes the notion of universe of sets.

The notion of <u>elementary topos</u> goes back to Bill Lawvere and Myles Tierney. (Birthday conference). One success of topos theory is Paul Cohen's result that the Continuum Hypothesis is independent from the Zermelo-Fraenkel axioms of set theory. Paul Cohen got the Fields Medal for that.

<u>Ex</u> Let $\mathcal{C}$ be any category. Then $\mathrm{Sets}^{\mathcal{C}}$ = cat. of functors $\mathcal{C} \to \mathrm{Sets}$ and natural transformations is a topos.

<u>The Topos $\mathrm{Sets}^{\mathcal{G}}$</u>

$\mathrm{Sets}^{\mathcal{G}}$ is a topos by the last example.

$\Omega := \{B \mid B \subseteq \mathcal{G} \text{ is a left ideal}\}$ is the subobject classifier w/ $\mathcal{G}$-action

$$\omega : \mathcal{G} \times \Omega \to \Omega$$
$$(m, B) \longmapsto \{n \in \mathcal{G} \mid n \circ m \in B\}$$

Thus, for any $\mathcal{G}$-set $\mu : \mathcal{G} \times X \to X$ there is a bijection

$$\{A \mid A \ \mathcal{G}\text{-subset of } X\} \longleftrightarrow \{\chi : X \to \Omega \mid \chi \ \mathcal{G}\text{-equivariant map}\}$$

$$\{x \in X \mid \chi(x) = \mathcal{G}\} \longleftarrow\!\!\shortmid\; \chi$$

$$A \longmapsto \Big(x \longmapsto \overbrace{\{T \in \mathcal{G} \mid \mu(T, x) \in A\}}^{\Omega \ni}\Big)$$

Prop (Noll) The left ideals in $\mathcal{G}$ are

$$\begin{tikzcd}[column sep=small]
 & \mathcal{G} \arrow[d, dash] & \\
 & \mathcal{P} := \mathcal{G} \setminus \{e\} \arrow[dl, dash] \arrow[dr, dash] & \\
\mathcal{L} := \{a, b, c, f, f^2\} \arrow[dr, dash] & & \mathcal{R} := \{a, b, c, g, g^2\} \arrow[dl, dash] \\
 & \mathcal{E} = \{a, b, c\} \arrow[d, dash] & \\
 & \emptyset = \emptyset. &
\end{tikzcd}$$

<u>truth</u> <u>values</u>

$$\{\mathcal{G}\} \longrightarrow \underset{\Omega}{\underset{\parallel}{\{\emptyset, \mathcal{E}, \mathcal{L}, \mathcal{R}, \mathcal{P}, \mathcal{G}\}}}$$ is the universal monomorphism in $\mathbf{Sets}^{\mathcal{G}}$.

Prop (Noll) The 22 $\mathcal{G}$-subsets of Ω are listed in terms of their characteristic morphisms χ. We single out the characteristic morphisms of the $\mathcal{G}$-subsets of Ω generated by $\mathcal{P}$, $\mathcal{L}$, and $\mathcal{R}$ respectively.

$$j_{\mathcal{P}} := \chi_{\{\mathcal{P}, \mathcal{G}\}} : \Omega \to \Omega \qquad j_{\mathcal{R}} := \chi_{\{\mathcal{R}, \mathcal{P}, \mathcal{G}\}} : \Omega \longrightarrow \Omega$$

$$j_{\mathcal{L}} := \chi_{\{\mathcal{L}, \mathcal{P}, \mathcal{G}\}} : \Omega \longrightarrow \Omega$$

Affine $\mathcal{G}$-actions on $\mathbb{Z}_{12}$ and Subactions

Here are some objects and subobjects in the topos $\mathbf{Sets}^{\mathcal{G}}$.

Recall $\mathcal{G}$ is generated by $f = {}^{1}3 : z \longmapsto 3z+1$

and $g = {}^{4}8 : z \longmapsto 8z+4$.

Let $\mu[m,n] : \mathcal{G} \times \mathbb{Z}_{12} \to \mathbb{Z}_{12}$ denote the

action with $\mu[m,n](f, z) = {}^{m}3(z)$

$\mu[m,n](g, z) = {}^{n}8(z)$.

Thus $\mu[1,4]$ is the natural action of $\mathcal{G}$ on $\mathbb{Z}_{12}$.

$\mu[1,4]$ restricts to an action $\mathcal{G} \times \{0,1,4\} \to \{0,1,4\}$

$\boxed{\text{Why?}}$ since $\mathcal{G}$ was defined to be precisely those affine transformations that preserve $\{0,1,4\}$.

Thus $\{0,1,4\}$ is a $\mathcal{G}$-subset of $\mathbb{Z}_{12}$ under action $\mu[1,4]$.

${}^{0}10(\{0,1,4\}) = \{0,10,4\}$ = "stretched triad"

Another action of interest is $\mu[10,4] : \mathcal{G} \times \mathbb{Z}_{12} \to \mathbb{Z}_{12}$, which restricts to $\mathcal{G} \times \{0,10,4\} \longrightarrow \{0,10,4\}$.

Interesting Morphisms $\mathbb{Z}_{12} \to \Omega$ and What They Classify

χ	$\chi^{-1}(ay)$
$j_P \circ \chi_{\{0,1,4\}}$	$\{0,1,4,9\} = \{C, E, G, E\flat\}$ major-minor mix
$j_L \circ \chi_{\{0,1,4\}}$	$\{0,1,4,5,8,9\}$ hexatonic system
$j_R \circ \chi_{\{0,1,4\}}$	$\{0,1,3,4,6,7,9,10\}$ octatonic system
$j_P \circ \chi_{\{0,10,4\}}$	$\{0,4,6,10\}$ French augmented sixth
$j_L \circ \chi_{\{0,10,4\}}$	$\{0,2,4,6,8,10\}$ wholetone system
$j_R \circ \chi_{\{0,10,4\}}$	$\{0,1,3,4,6,7,9,10\}$ octatonic system
$j_P \circ \chi_{\{3,1,7\}}$	
$j_L \circ \chi_{\{3,1,7\}}$	
$j_R \circ \chi_{\{3,1,7\}}$	

(using circle of semitones encoding).

"Modalities where respective triad is locally present"

Musical Interpretations of Topos Sets of $\mathcal{T}$ in Skrjabin, Op 65, #3

Each left hand chord is partly of the form

$\{0+k,\ 10+k,\ 4+k\}$ which is a subobject of

$$\mu[10-2k,\ 4-7k] : \mathcal{T} \times \mathbb{Z}_{12} \longrightarrow \mathbb{Z}_{12},$$

so we can consider its characteristic function and evaluate that characteristic function on the bass note and other notes in the cell. This is the table at the end of the article.

Ex Calculate for first cell $\{G, F, B\} = \{1, 11, 5\}$

$$\chi_{\{1,11,5\}}(\underset{G}{\overset{}{1}}) = \mathcal{T}$$

$$\chi_{\{1,11,5\}}(\underset{A}{3}) = L$$

$$\chi_{\{1,11,5\}}(\underset{4.}{E}) = R.$$

Observe — lowest truth value $\emptyset$ is entirely absent in local comparisons

- $\chi_{\{1,5,11\}}(7) = P$ = largest nontrue truth value

$7 = $ Db = tritone above bass in first measure

The continuous tritone pendulum is as close to "true" as possible.

- The first bass notes G, Db, A $= 1, 7, 3 = 3, 1, 7 = 0+3, 10+1, 4+3$

are also a stretched triad.

Have students check which notes are missing from upgrades of $\{3,1,7\}$

- This large-scale stretched triad is always locally present in bass, since bass is always in the sets corresponding to $J_{\mathcal{O}} \circ \chi_{\{3,1,7\}}$, $J_{\mathcal{R}} \circ \chi_{\{3,1,7\}}$, $J_{\mathcal{I}} \circ \chi_{\{3,1,7\}}$

(Skrjabin avoids in bass exactly $2 = D$ and $8 = Ab$, the only two notes not in an upgrade of $\{3,7,13\}$)

- The lowest truth value $\mathcal{E}$ for $\{3,1,7\}$ (only $2 = D$, $8 = Ab$ satisfy $\chi_{\{3,7,13\}} = \mathcal{E}$) is also globally absent, so global behavior of bass w.r.t. $\{3,7,13\}$ mimics local behavior (of first point.)