# Mathematics and Music
# Lecture 1

Thomas M. Fiore

fiore@math.uchicago.edu

http://www.math.uchicago.edu/~fiore/

# Introduction

Mathematics is a very powerful descriptive tool in the physical sciences.

Similarly, musicians use mathematics to communicate ideas about music.

In these lectures, we will discuss the mathematics commonly used by musicians.

# What is Music Theory?

- David Hume: impressions become tangible and form ideas.
- Music theory supplies us with conceptual categories to organize and understand music.
- In other words, music theory provides us with the means to find a good way of hearing a work of music.

# Who Needs Music Theory?

- Composers
- Performers
- Listeners

# Directions in U.S. Mathematical Music Theory

- Scale Theory (Lecture 1)
  (Carey, Clampitt, Clough, Douthett...)

- Set Theory (Lecture 2)
  (Forte, Morris, Perle, Rahn...)

- Transformational Analysis (Lectures 3, 4)
  (Cohn, Lewin, Satyendra...)

# Scale Theory

Topics:
generation of scales, properties of scales, classification of scales

Mathematical Tools:
naive set theory, combinatorics, number theory, continued fractions

# Set Theory

Topics:
analyze music via collections of pitches and pitch classes, transposition and inversion, especially fruitful for twentieth century music

Mathematical Tools:
naive set theory (not mathematical set theory!), group theory, combinatorics

# Transformational Theory

Topics:
Which transformations are idiomatic for a piece?

Mathematical Tools:
group actions on sets, topology

# Basics

- Pitch and Pitch Class
- Integer Model of Pitch
- Arithmetic Modulo 12
- $\mathbb{Z}_{12}$ Model of Pitch Classes
- Intervals
- Transposition
- Inversion

# Pitch and Pitch Class

A *pitch* is a single sound at a distinguishable frequency.

If $a$ and $b$ are pitches designated by their frequency, we write

$$a \sim b$$

if $a/b = 2^j$ for some $j \in \mathbb{Z}$, in other words if $a$ and $b$ are a whole number of *octaves* apart.

This is an equivalence relation, and the equivalence classes are called *pitch classes*.

# The Integer Model of Pitch

We assume *equal tempered tuning*, in other words, the octave is broken up into *twelve* equal intervals. This is irrational.

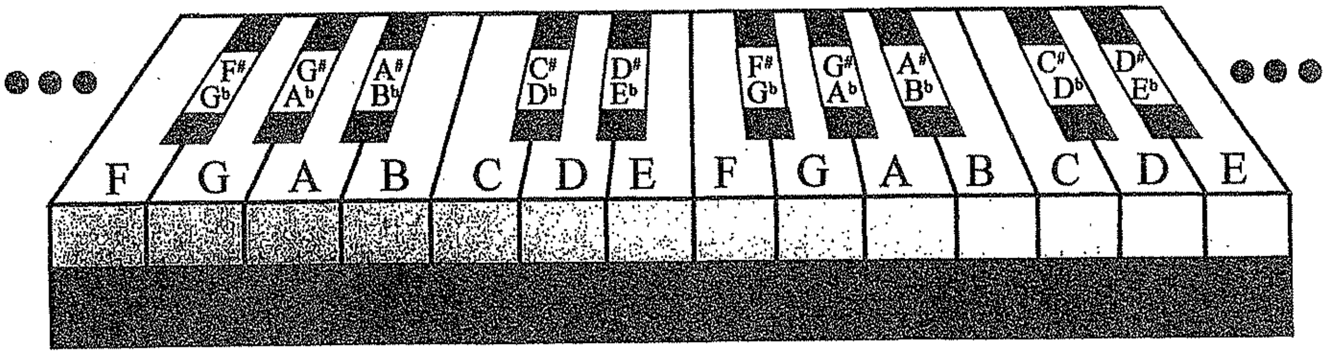
Middle $A$ has 440 Hz.

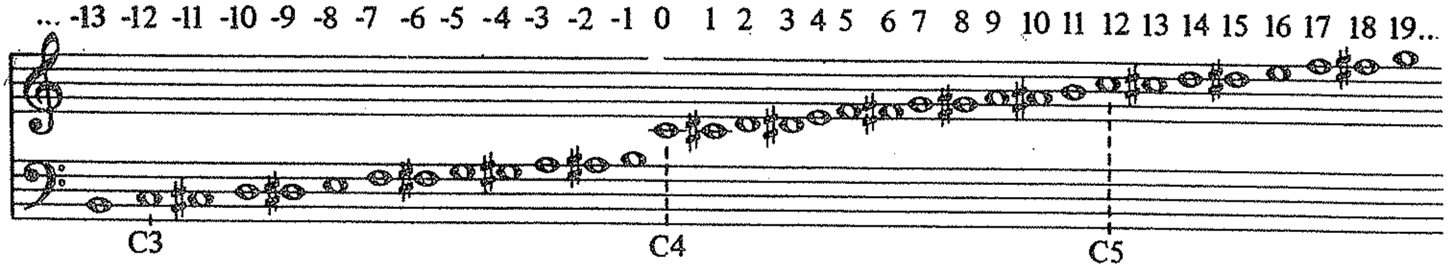
Middle $C$ is assigned the number 0, the next higher pitch 1, and the next higher 2, etc. The pitch just below middle $C$ is -1, just below that is -2 etc.

$<$ in $\mathbb{Z}$ corresponds to *lower than* in pitch.

Two pitches belonging to the same *pitch class* have the same letter name. In fact, we use letter names to indicate *pitch classes*.

$\Rightarrow$ Arithmetic Modulo 12


F# Gb
G# Ab
A# Bb
C# Db
D# Eb
F# Gb
G# Ab
A# Bb
C# Db
D# Eb
F G A B C D E F G A B C D E


... -13 -12 -11 -10 -9 -8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19...
C3
C4
C5

# Arithmetic Modulo 12

Think of a clock with 0 in the 12 o'clock position.

1+2=3 mod 12

11 +1 =0 mod 12

11+2 = 1 mod 12

11+ 5 = 4 mod 12

$\mathbb{Z}_{12} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$

# The $\mathbb{Z}_{12}$ Model of Pitch Class

Since we are using equal tempered tuning, we can use *enharmonic equivalence*.

$$C = 0$$
$$C\sharp = D\flat = 1$$
$$D = 2$$
$$D\sharp = E\flat = 3$$
$$E = 4$$
$$F = 5$$
$$F\sharp = G\flat = 6$$
$$G = 7$$
$$G\sharp = A\flat = 8$$
$$A = 9$$
$$A\sharp = B\flat = 10$$
$$B = 11$$

# Intervals

We have a *chromatic interval function*.

$$int : \mathbb{Z}_{12} \times \mathbb{Z}_{12} \rightarrow \mathbb{Z}_{12}$$

$$int(a, b) := b - a$$

| Interval | Name |
|---|---|
| 7 | Perfect Fifth |
| 5 | Perfect Fourth |
| 4 | Major Third |
| 3 | Minor Third |
| 1 | semitone, half step |
| 2 | whole tone, whole step |
| 6 | Tritone |

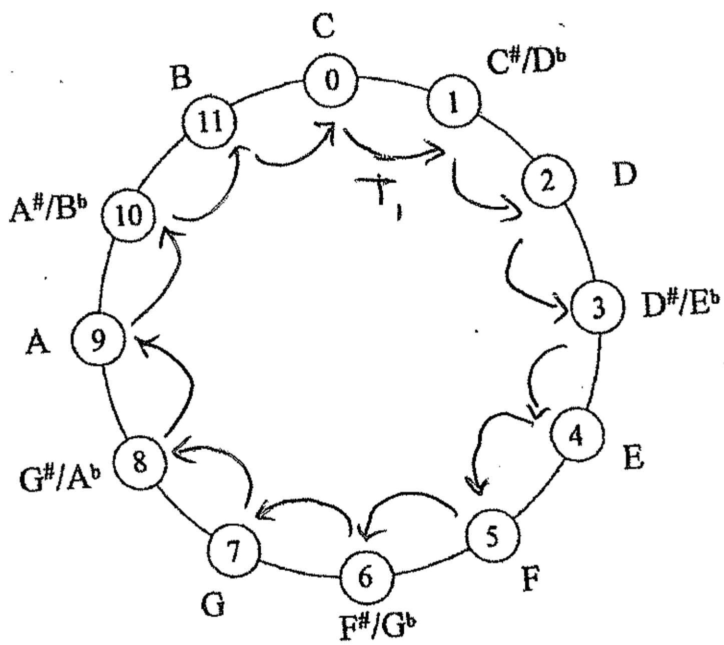
# Transposition

The bijective function

$$T_n : \mathbb{Z}_{12} \to \mathbb{Z}_{12}$$

$$T_n(x) := x + n$$

is called *transposition* by musicians, or *translation* by mathematicians.




$T_1(0) = 1$, $T_1(11) = 0$, $T_7(3) = 10$, $T_7(7) = 2$

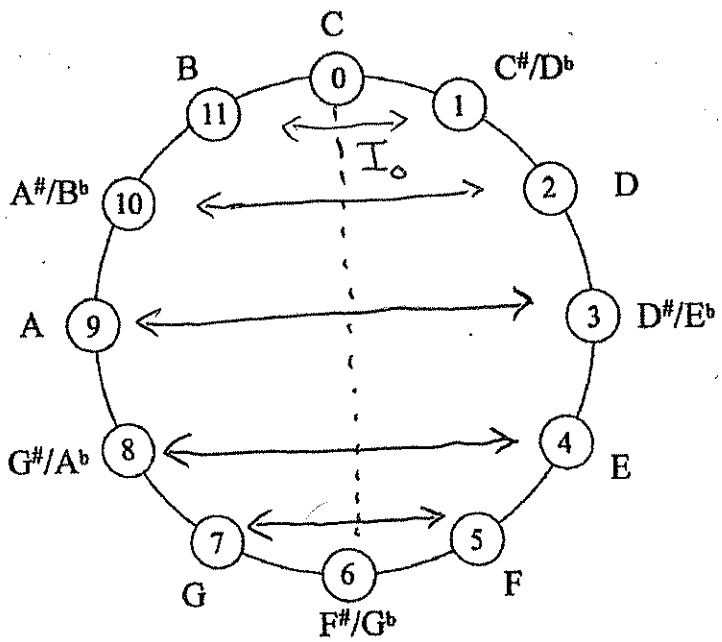
# Inversion

The bijective function

$$I_n : \mathbb{Z}_{12} \to \mathbb{Z}_{12}$$

$$I_n(x) := -x + n$$

is called *inversion* by musicians, or *reflection* by mathematicians.




$I_0(1) = 11$, $I_0(5) = 7$, $I_7(1) = 6$, $I_7(5) = 2$

# Scale Theory

A *scale* is a subset of $\mathbb{Z}_{12}$. Sometimes *scale* means the transposition class of a subset.

A scale is the pitch material that a composer draws on in a passage or work of music.

The most common scales are the major scale, the minor scale, the pentatonic scale, the octatonic scale, the whole tone scale, and the chromatic scale.

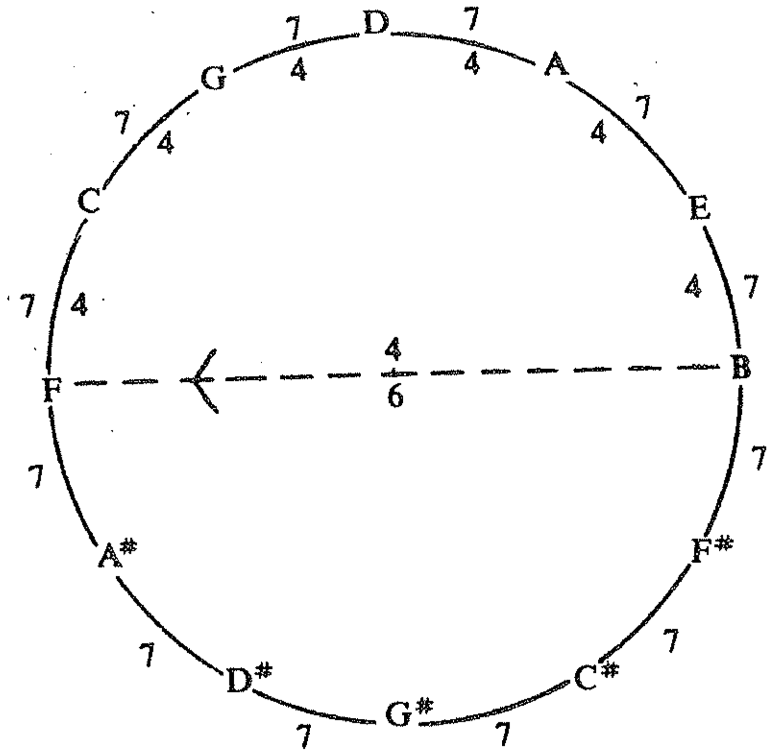
We introduce these and study their properties.

# The Major Scale

Consider repeated applications of $T_7$ to $F = 5$.

$$5, 0, 7, 2, 9, 4, 11, 6, 1, 8, 3, 10, 5$$

$$F, C, G, D, A, E, B, F\sharp, C\sharp, G\sharp, D\sharp, A\sharp, F$$




This is the *circle of fifths.*

# The Major Scale

The *$C$-major scale* consists of the first seven pitch classes in the circle of fifths (the white keys on the piano).

$$\{C, D, E, F, G, A, B, C\}$$

$$\{0, 2, 4, 5, 7, 9, 11, 0\}$$

The *step intervals* are 2-2-1-2-2-2-1. Any scale with these step intervals is called a *major scale* or *diatonic scale.*

The $G$-major scale consists of the seven pitch classes in the circle of fifths beginning on $C$.

**Theorem 1** *The major scale is the unique scale of cardinality seven with the largest number of perfect fifths.*

# The Minor Scale

As a set, a *minor scale* is the same as a major scale. But the starting point is different. The *$A$-minor scale* is

$$\{A, B, C, D, E, F, G, A\}$$

$$\{9, 11, 0, 2, 4, 5, 7, 9\}.$$

The step intervals are 2-1-2-2-1-2-2. Any scale with these step intervals is called a *minor scale*.

major=cheerful
minor=sad, mysterious

**Example 1** *Schumann, "The Happy Peasant" is in $F$-major. Schumann, "The Wild Horseman" is in $A$-minor, and then $F$-major.*

# The Pentatonic Scale

The *$F$-major pentatonic scale* consists of the first five pitch classes of the circle of fifths.

$$\{F, G, A, C, D, F\}$$

$$\{5, 7, 9, 0, 2, 5\}$$

The step intervals are 2-2-3-2-3. Any scale with these step intervals is called a *pentatonic scale.*

The complement of a major scale is a pentatonic scale, for example the black keys form a pentatonic scale.

**Example 2** *Indonesian Gamelan music, blues, spirituals, Amazing Grace, blues, Bartók, Debussy*

# The Octatonic Scale

The $D\flat$ *octatonic scale* or $D\flat$ *diminished scale* is

$$\{D\flat, E\flat, E, F\sharp, G, A, B\flat, C, D\flat\}$$

$$\{1, 3, 4, 6, 7, 9, 10, 0, 1\}.$$

The step intervals are 2-1-2-1-2-1-2-1. Any scale with these step intervals is called an *octatonic scale.*

There are only three octatonic scales.

**Example 3** *Bartok, Rimsky-Korsakov, Scriabin, Stravinsky, Jazz*

# The Whole Tone Scale

The *whole tone scale on $C$* is

$$\{C, D, E, F\sharp, G\sharp, A\sharp, C\}$$

$$\{0, 2, 4, 6, 8, 10, 0\}.$$

The step intervals are 2-2-2-2-2-2. Any scale with these step intervals is called a *whole tone scale*.

There are only two whole tone scales.

**Example 4** *19th century Russian music, Bartok, Debussy, impressionist composers, Jazz, Thelonius Monk*

# The Chromatic Scale

The *chromatic scale* is

$$\{C, C\sharp, D, D\sharp, E, F, F\sharp, G, G\sharp, A, A\sharp, B, C\}$$

$$\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 0\}.$$

**Example 5** *Late 19th century music such as Wagner, 20th century atonal music such as Schoenberg, Berg, Webern, Krenek, Carter*

# Scale Property: Generated

A scale is said to be *generated* if it is obtained by an iteration of $T_n$ for some $n$.

**Example 6** *The major scale is generated by the interval of a perfect fifth by repeated applications of* $T_7$.

$$\{5, 0, 7, 2, 9, 4, 11\}$$

**Example 7** *The pentatonic scale is generated by the interval of a perfect fifth by repeated applications of* $T_7$.

$$\{5, 0, 7, 2, 9\}$$

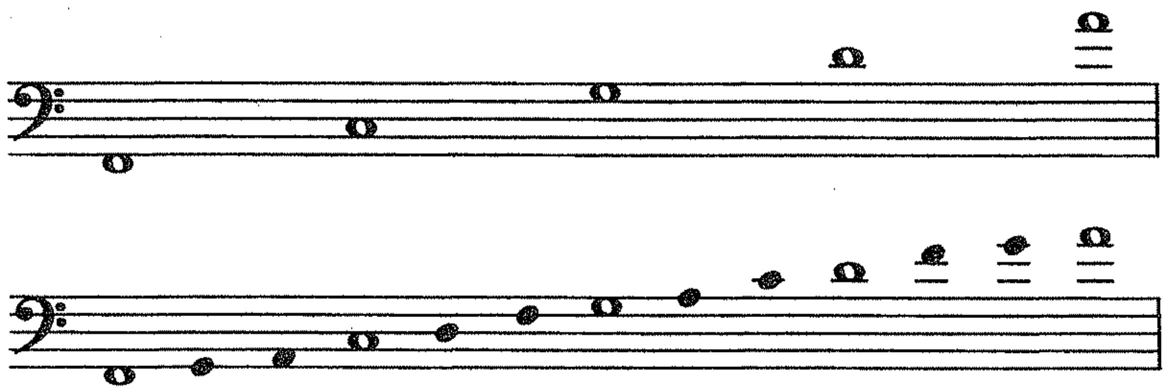
**Example 8** *The octatonic scale is not generated.*

$$\{0, 1, 3, 4, 6, 7, 9, 10\}$$

# Scale Property: Well Formed

A generated scale is said to be *well formed* if each generating interval spans the same number of scale steps.

**Example 9** *The pentatonic scale is well formed.*



The pentatonic scale, generated by the perfect fifth, is well-formed since all perfect fifths span the same number of steps (3).

EXAMPLE 1

# Scale Property: Myhill Property

A scale is said to have the *Myhill Property* if each scale interval comes in two chromatic sizes.

**Example 10** *The major scale satisfies the Myhill Property.*

$$\{0, 2, 4, 5, 7, 9, 11, 0\}$$

*That is why we have major/minor seconds, major/minor thirds, etc.*

**Theorem 2** *(Carey, Clampitt) A non-chromatic, generated scale in $\mathbb{Z}_{12}$ satisfies the Myhill Property if and only it is well formed.*

# Scale Property: Maximally Even

A scale is said to be *maximally even* if each scale intervals comes in either one chromatic size or two chromatic sizes of consecutive integers.

**Example 11** *The octatonic scale is maximally even.*

$$\{0, 1, 3, 4, 6, 7, 9, 10, 0\}$$

| *Scale Interval* | *Chromatic Sizes* |
|---|---|
| *1* | *1,2* |
| *2* | *3* |
| *3* | *4,5* |
| *4* | *6* |

# Uniqueness of the Major Scale, Classification and Enumeration

**Theorem 3** *(Clough, Engebretsen, Kochavi) The major scale is uniquely characterized by eight properties.*

**Theorem 4** *(Clough, Engebretsen, Kochavi) Scales have been classified according to these eight properties. There are concrete algorithms that enumerate all scales satisfying any subset of the eight properties.*

# Self Similarity

The major scale is *self similar*: any scale interval contains (approximately) the same number of step intervals with chromatic interval 1 as the entire scale itself, namely 2/7.

| span | smaller | no. of m2s | no. of M2s | larger | no. of m2s | no. of M2s |
|---|---|---|---|---|---|---|
| 1 | m2 | 1 | 0 | M2 | 0 | 1 |
| 2 | m3 | 1 | 1 | M3 | 0 | 2 |
| 3 | P4 | 1 | 2 | A4 | 0 | 3 |
| 4 | d5 | 2 | 2 | P5 | 1 | 3 |
| 5 | m6 | 2 | 3 | M6 | 1 | 4 |
| 6 | m7 | 2 | 4 | M7 | 1 | 5 |

EXAMPLE 3: DECOMPOSITION OF DIATONIC INTERVALS INTO STEP INTERVALS

| Span | | Larger | | | | Smaller |
|---|---|---|---|---|---|---|
| 1 | 2nds | 0/1 | < | 2/7 | < | 1/1 |
| 2 | 3rds | 0/2 | < | 2/7 | < | 1/2 |
| 3 | 4ths | 0/3 | < | 2/7 | < | 1/3 |
| 4 | 5ths | 1/4 | < | 2/7 | < | 2/4 |
| 5 | 6ths | 1/5 | < | 2/7 | < | 2/5 |
| 6 | 7ths | 1/6 | < | 2/7 | < | 2/6 |

EXAMPLE 4: DISTRIBUTION OF HALF STEPS IN DIATONIC INTERVALS COMPARED WITH HALF STEPS PER OCTAVE

# Self Similarity and Other Properties

**Theorem 5** *(Carey, Clampitt) A non-chromatic, generated scale in* $\mathbb{Z}_{12}$ *is self similar*
*if and only if it satisfies the Myhill property*
*if and only if it is well-formed.*

# Summary

- In this talk I have introduced some of the conceptual categories that music theorists use to make aural impressions into tangible ideas in the sense of Hume.

- These conceptual categories include $\mathbb{Z}_{12}$, the functions $T_n$ and $I_n$, and scale properties.

- We have used these tools in our understanding of Schumann, Debussy, and Bartok.