# Mathematics and Music Lecture 2

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# Set Theory

Topics:
analyze music via collections of pitches and pitch classes, transposition and inversion, especially fruitful for twentieth century music

Mathematical Tools:
naive set theory (not mathematical set theory!), group theory, combinatorics

Today we focus on Set Theory.

# History

In 1907, something revolutionary happened: Schoenberg broke with tonality. This means that he wrote music outside of the major and minor scales. Instead he used the entire chromatic universe $\mathbb{Z}_{12}$, and intentionally avoided reference to any major or minor scale.

This was the conclusion of a centuries long progression: the erosion of tonality. Modulations became more and more common, so that the notion of "key" was weakened.

Tonal methods of analysis were not so successful with twentieth century music. Set theory was developed as a tool to analyze the new music.

# The $\mathbb{Z}_{12}$ Model of Pitch Class

Set theory typically uses pitch classes instead of pitches. So two pitches are considered to be equal if they are an octave apart, in other words if they have the same letter name. Hence, set theorists use $\mathbb{Z}_{12}$ as below.

$$C = 0$$
$$C\sharp = D\flat = 1$$
$$D = 2$$
$$D\sharp = E\flat = 3$$
$$E = 4$$
$$F = 5$$
$$F\sharp = G\flat = 6$$
$$G = 7$$
$$G\sharp = A\flat = 8$$
$$A = 9$$
$$A\sharp = B\flat = 10$$
$$B = 11$$

# Set Theoretic Operations

Transposition:

$$T_n : \mathbb{Z}_{12} \to \mathbb{Z}_{12}$$

$$T_n(x) := x + n$$

$$T_n(\langle x_1, x_2, \ldots, x_m \rangle) := \langle T_n(x_1), T_n(x_2), \ldots, T_n(x_m) \rangle$$

Inversion:

$$I_n : \mathbb{Z}_{12} \to \mathbb{Z}_{12}$$

$$I_n(x) := -x + n$$

$$I_n(\langle x_1, x_2, \ldots, x_m \rangle) := \langle I_n(x_1), I_n(x_2), \ldots, I_n(x_m) \rangle$$

Retrograde:

$$R(\langle x_1, x_2, \ldots, x_m \rangle) := \langle x_m, x_{m-1}, \ldots, , x_1 \rangle$$

*Set* means subset of $\mathbb{Z}_{12}$.

# Interval Multiset / Interval Vector

Another tool used by set theorists is the interval multiset, or equivalently the interval vector.

If $A \subseteq \mathbb{Z}_{12}$, then the *interval multiset of* $A$ is the multiset with an entry $min\{a - b, b - a\}$ for each distinct pair $a, b \in A$.

**Example 1** *The interval multiset of* $\{1, 2, 3\}$ *is* $\{1, 1, 2\}$. *Equivalently, the interval vector of* $\{1, 2, 3\}$ *is* $\langle 2, 1, 0, 0, 0, 0 \rangle$.

**Example 2** *The interval multiset of* $\{1, 2, 7, 10\}$ *is* $\{1, 3, 3, 4, 5, 6\}$. *Equivalently the interval vector of* $\{1, 2, 7, 10\}$ *is* $\langle 1, 0, 2, 1, 1, 1 \rangle$.

## $T_n, I_n$ Preserve Interval Content

$T_n(a) - T_n(b) = a + n - (b + n) = a - b$
$T_n(b) - T_n(a) = b + n - (a + n) = b - a$

$I_n(a) - I_n(b) = -a + n - (-b + n) = -a + b$
$I_n(b) - I_n(a) = -b + n - (-a + n) = b - a$

**Lemma 1** *If $A \subset \mathbb{Z}_{12}$, then $T_n(A)$ and $I_n(A)$ and $A$ all have the same interval multiset. Equivalently, they have the same interval vector.*

Because of this audible similarity, any two sets that become equal after transposition or inversion are considered *the same.*

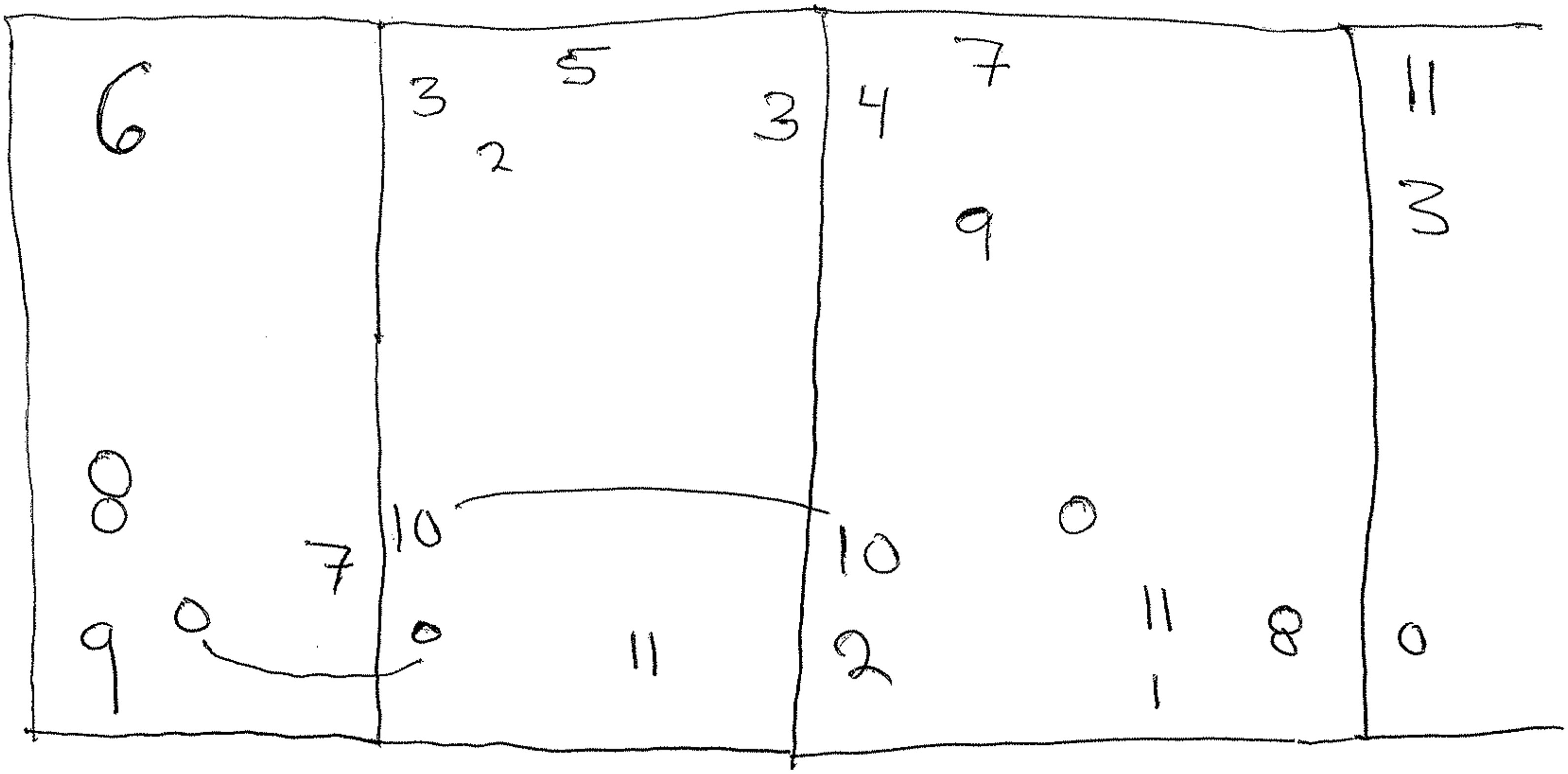
In fact, this is an equivalence relation on the set of subsets of $\mathbb{Z}_{12}$.

# The Hexachord Theorem

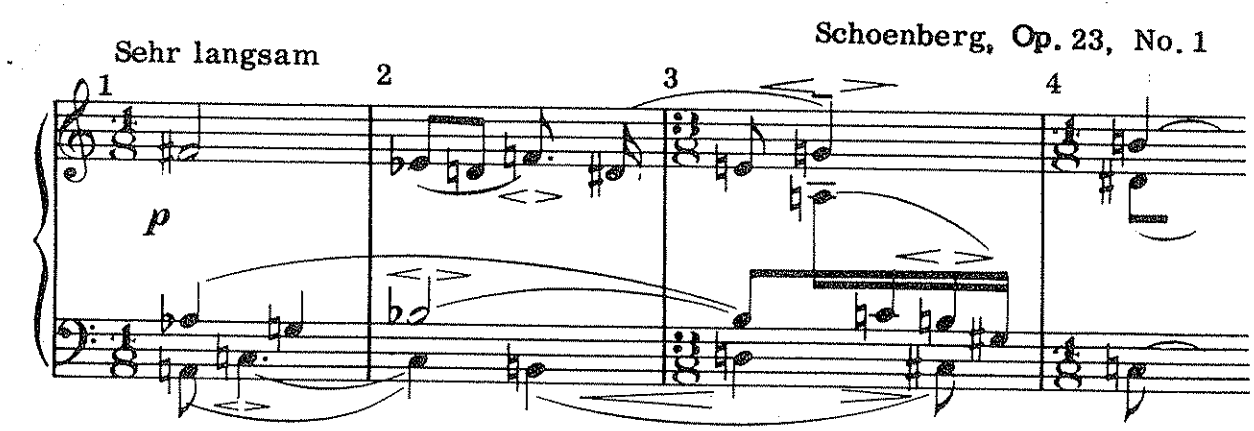
**Theorem 2** *If* $A \subseteq \mathbb{Z}_{12}$ *has six elements, then the interval vector of* $A$ *is the same as the interval vector of its complement* $\overline{A} = \mathbb{Z}_{12} \setminus A$

This was empirically observed by Schoenberg: the two halves of a tone row are complimentary hexachords.

See Klau in *Mathematics Magazine* for a short proof.




Schoenberg, Op. 23, Number 1, Measures 1–4.




See George Perle, Serial Composition and Atonality.

# Schoenberg Opus 23, Number 1

The cell $\langle A\flat, G, B\flat \rangle = \langle 8, 7, 10 \rangle$ appears in many guises in measures 1-4.

| Transposed Forms | Inverted Forms |
|---|---|
| $\langle 8, 7, 10 \rangle$ | $\langle 4, 5, 2 \rangle$ |
| $\langle 9, 8, 11 \rangle$ | $\langle 5, 6, 3 \rangle$ |
| $\langle 10, 9, 0 \rangle$ | $\langle 6, 7, 4 \rangle$ |
| $\langle 11, 10, 1 \rangle$ | $\langle 7, 8, 5 \rangle$ |
| $\langle 0, 11, 2 \rangle$ | $\langle 8, 9, 6 \rangle$ |
| $\langle 1, 0, 3 \rangle$ | $\langle 9, 10, 7 \rangle$ |
| $\langle 2, 1, 4 \rangle$ | $\langle 10, 11, 8 \rangle$ |
| $\langle 3, 2, 5 \rangle$ | $\langle 11, 0, 9 \rangle$ |
| $\langle 4, 3, 6 \rangle$ | $\langle 0, 1, 10 \rangle$ |
| $\langle 5, 4, 7 \rangle$ | $\langle 1, 2, 11 \rangle$ |
| $\langle 6, 5, 8 \rangle$ | $\langle 2, 3, 0 \rangle$ |
| $\langle 7, 6, 9 \rangle$ | $\langle 3, 4, 1 \rangle$ |

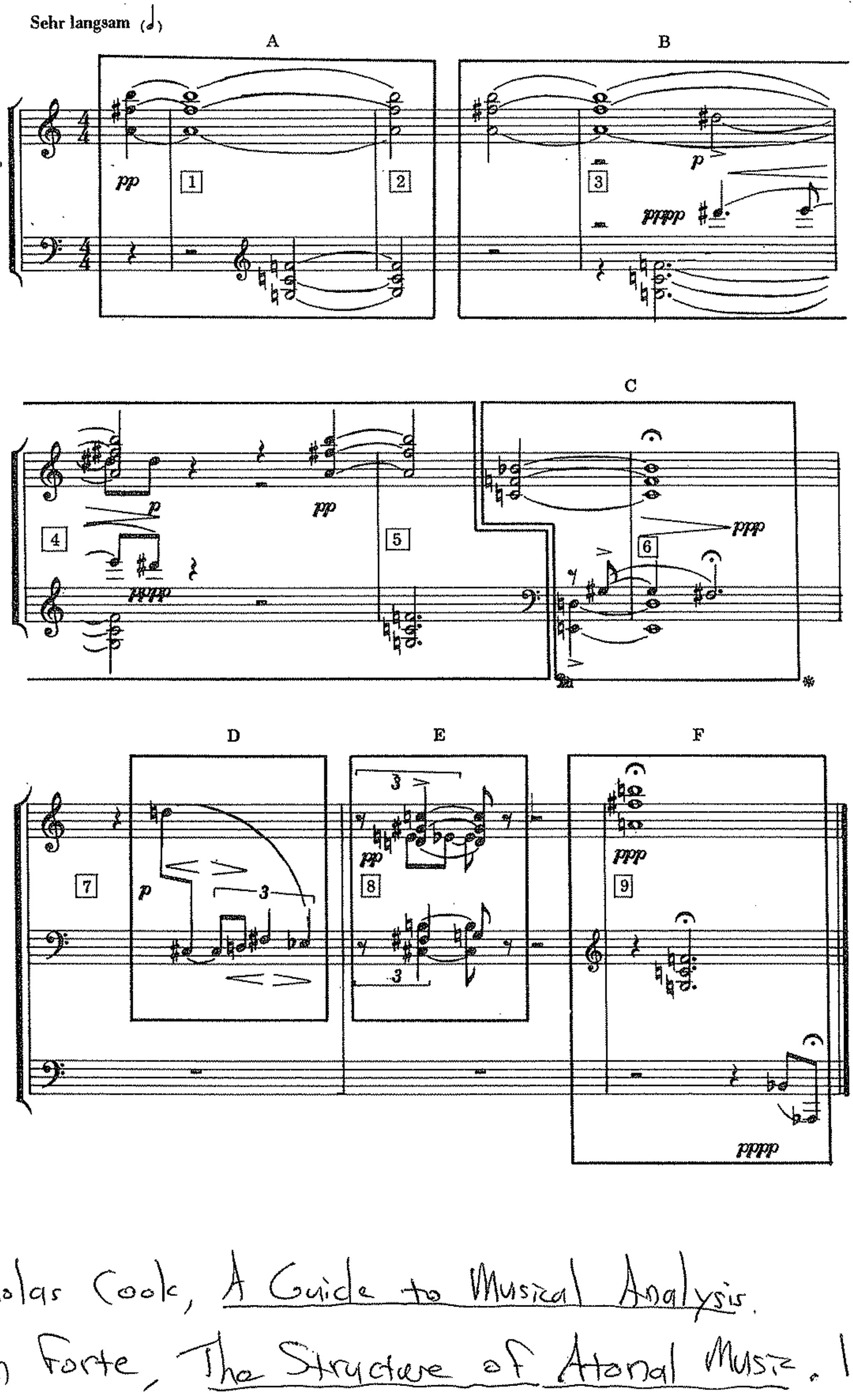
# Schoenberg Opus 23, Number 1

The retrograde of $\langle 8, 7, 10\rangle$ is $\langle 10, 7, 8\rangle$.

| Transposed Forms | Inverted Forms |
|---|---|
| $\langle 10, 7, 8\rangle$ | $\langle 2, 5, 4\rangle$ |
| $\langle 11, 8, 9\rangle$ | $\langle 3, 6, 5\rangle$ |
| $\langle 0, 9, 10\rangle$ | $\langle 4, 7, 6\rangle$ |
| $\langle 1, 10, 11\rangle$ | $\langle 5, 8, 7\rangle$ |
| $\langle 2, 11, 0\rangle$ | $\langle 6, 9, 8\rangle$ |
| $\langle 3, 0, 1\rangle$ | $\langle 7, 10, 9\rangle$ |
| $\langle 4, 1, 2\rangle$ | $\langle 8, 11, 10\rangle$ |
| $\langle 5, 2, 3\rangle$ | $\langle 9, 0, 11\rangle$ |
| $\langle 6, 3, 4\rangle$ | $\langle 10, 1, 0\rangle$ |
| $\langle 7, 4, 5\rangle$ | $\langle 11, 2, 1\rangle$ |
| $\langle 8, 5, 6\rangle$ | $\langle 0, 3, 2\rangle$ |
| $\langle 9, 6, 7\rangle$ | $\langle 1, 4, 3\rangle$ |

Consider also the forms of $\langle 6, 3, 2\rangle$.

**Fig. 53** Schoenberg, Op. 19/6, with segmentation




See Nicholas Cook, <u>A Guide to Musical Analysis</u>.
and Allen Forte, <u>The Structure of Atonal Music</u>. 1973.

# Schoenberg Opus 19, Number 6

The transpositional and inversional relationships between the underlying sets allow us to see the internal logic of the piece.

$A = \{C, F, F\sharp, G, A, B\} = \{0, 5, 6, 7, 9, 11\}$

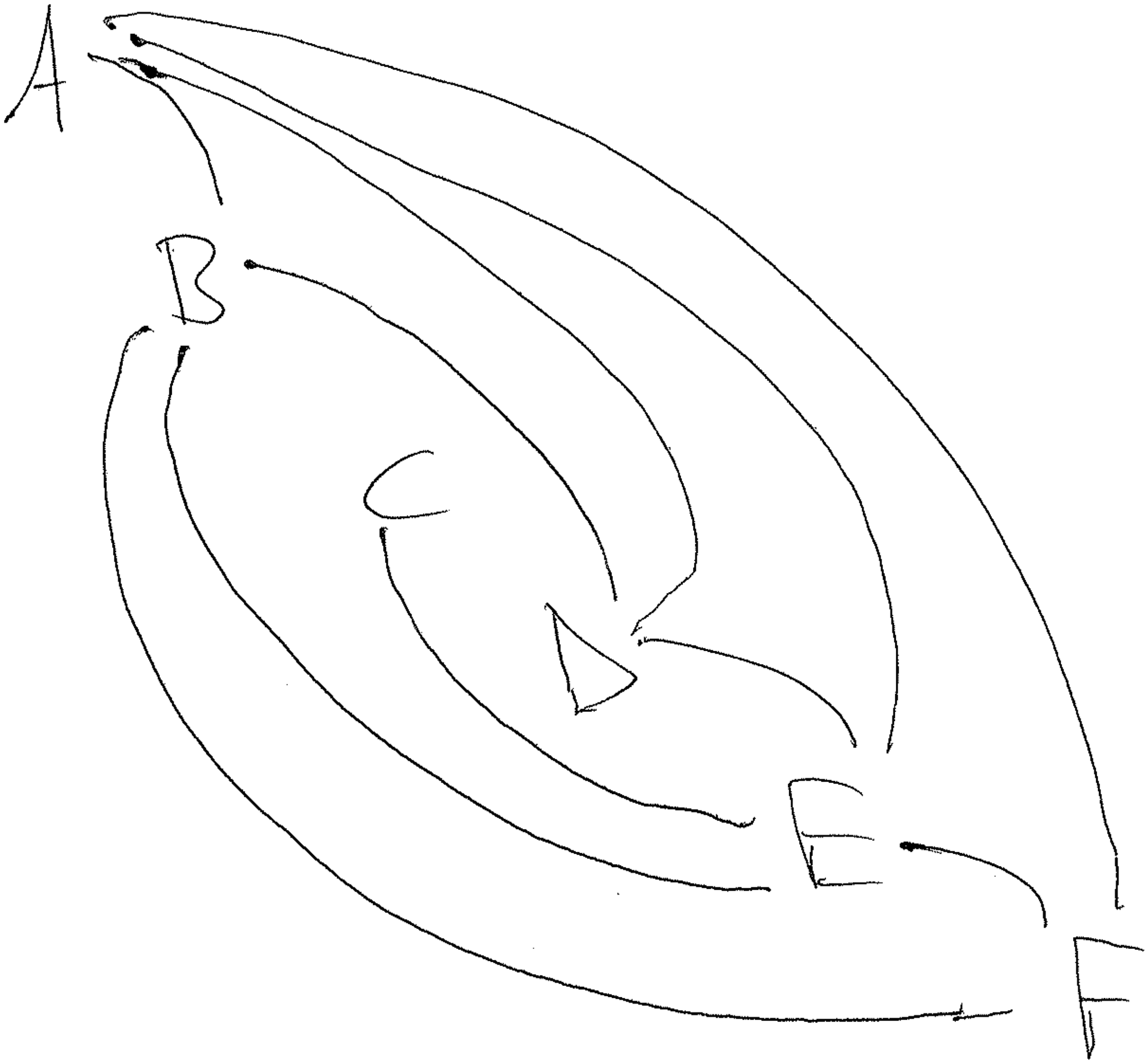
$B = \{C, D\sharp, E, F, F\sharp, G, A, B\} = \{0, 3, 4, 5, 6, 7, 9, 11\}$

$C = \{C, D, E, F, F\sharp, G\sharp, A\sharp\} = \{0, 2, 4, 5, 6, 8, 10\}$

$D = \{C\sharp, D, D\flat, F\sharp\} = \{1, 2, 3, 6\}$

$E = \{C, C\sharp, D, D\sharp, E, F\sharp, G, G\sharp, B\} =$
$\{0, 1, 2, 3, 4, 6, 7, 8, 11\}$

$F = \{C, F, F\sharp, G, G\sharp, A, A\sharp, B\} =$
$\{0, 5, 6, 7, 8, 9, 10, 11\}$




E connects to everything,

E is the source of the rest of the piece: surface features are there.