Mathematics and Music Lecture 3

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Transformational Theory

Topics:
Which transformations are idiomatic for a piece?
Neo-Riemannian Theory, Tonnetze, contextual inversions...

Mathematical Tools:
group actions on sets, topology

Today we begin our discussion of transformational theory, which we will continue next time.

History

Transformational Analysis has been developing for the past 20 years or so, and is currently a very active area of research. The foundational work

David Lewin. *Generalized Musical Intervals and Transformations*. New Haven: Yale University Press, 1987.

pioneered the field. Lewin introduced the fundamental notion of *generalized interval system*. Nevertheless, the roots of the subject extend back as far as Hugo Riemann 1849-1919.

Unlike set theory, Transformational Theory has been successfully used to analyze works of music from many periods.

Generalized Interval Systems

Definition 1 *(Lewin) A generalized interval system* $(S, IVLS, int)$ *consists of a musical space* S*, a group* $IVLS$ *of intervals, and an interval function*

$$int : S \times S \to IVLS$$

such that

1. For all $r, s, t \in S$*,*

$$int(r, s) * int(s, t) = int(r, t).$$

2. For each $s \in S$ *and* $i \in IVLS$ *there exists a unique* $t \in S$ *such that*

$$int(s, t) = i.$$

Example 1 $S := \mathbb{Z}_{12}, IVLS = \mathbb{Z}_{12}$*, and* $int(s, t) := t - s$ *is the basic example of a GIS in atonal theory.*

Transpositions

The most elementary example of a generalized interval system (GIS) is $(\mathbb{Z}_{12}, \mathbb{Z}_{12}, int)$. One of the salient features of this example is *transposition*

$$T_n : \mathbb{Z}_{12} \to \mathbb{Z}_{12}$$

$$T_n(x) := x + n.$$

In fact, transpositions can be defined in any GIS.

Definition 2 *Let $(S, IVLS, int)$ be a GIS and $i \in IVLS$. Then transposition by i is the unique function $T_i : S \to S$ such that*

$$int(s, T_i(s)) = i$$

for all $s \in S$.

Group Actions

If G is a group and X is a set, then a *(left) group action of* G *on* X is a function

$$G \times X \to X$$

such that

$$g_1(g_2x) = (g_1g_2)x$$

$$1_Gx = x$$

for all $g_1, g_2 \in G$ and $x \in X$.

Simple Transitivity

A group action is *transitive* if for any $x, y \in X$ there exists a $g \in G$ such that $gx = y$, in other words there is only one orbit and that orbit is all of X. A group action is *simple* or *free* if

$gx = g'x$ for some $x \in X \Rightarrow g = g'$.

Note that free is stronger than faithful. A group action is *simply transitive* if it is both transitive and simple. Such a thing is also known as a *torsor*. See John Baez's site, Week 234.

Example 2 *The group* $\{T_n : n \in \mathbb{Z}_{12}\}$ *acts simply transitively on* $\mathbb{Z}_{12}$.

GIS's and Simple Transitivity

Theorem 1 *If* $(S, IVLS, int)$ *is a generalized interval system, then its group of transpositions acts simply transitively on* S*.*

Example 3 *The transpositions in the GIS* $(\mathbb{Z}_{12}, \mathbb{Z}_{12}, int)$ *act simply transitively.*

Theorem 2 *Any simply transitive group action gives rise to a generalized interval system.*

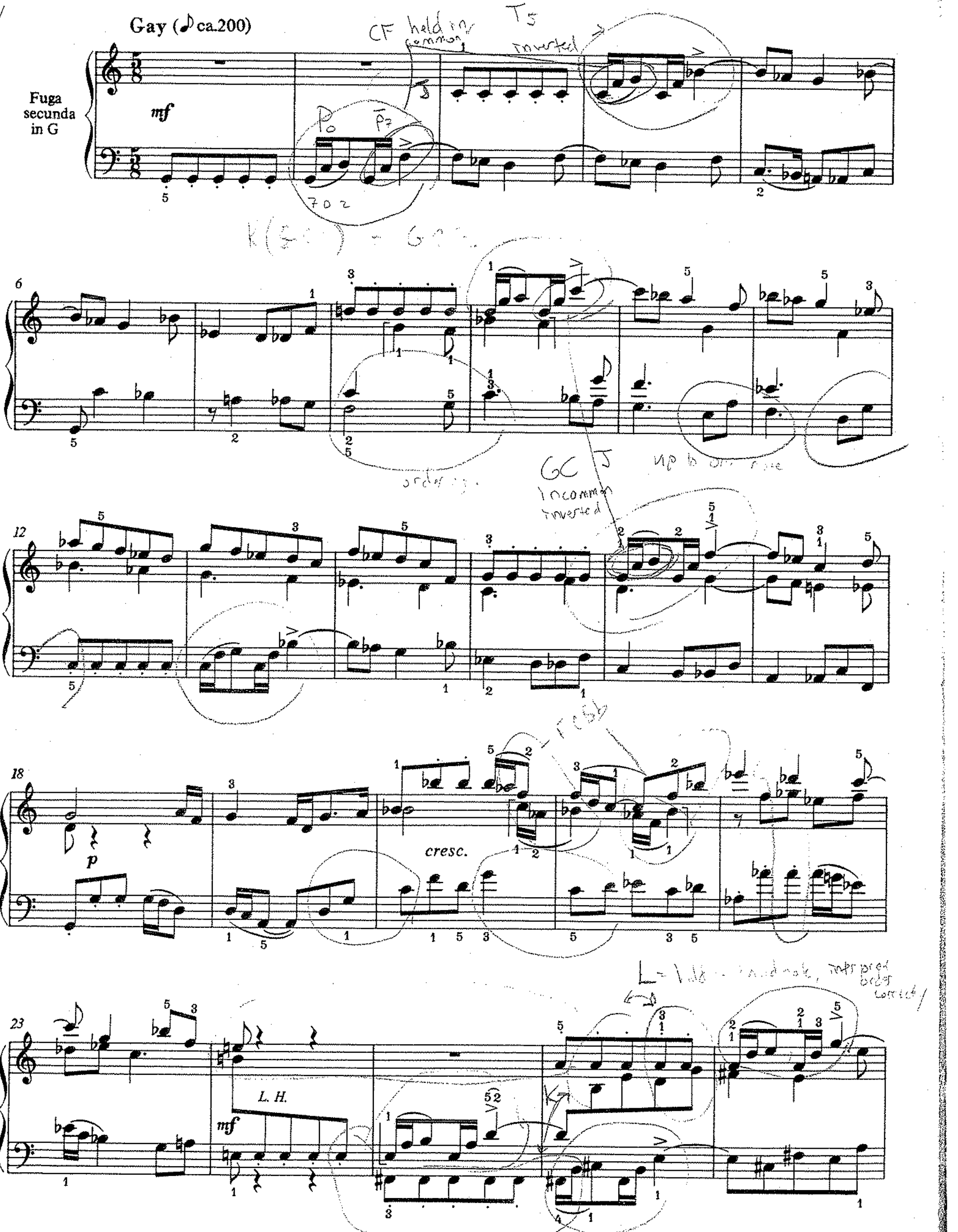
Thus, we have two ways of looking at a generalized interval system. We will use the latter.

Musical Example: Hindemith, *Ludus Tonalis*, Fugue in G

Q. Which transformations are idiomatic for this piece?
A. Contextual inversion J and transpositions T_2, T_3, T_5, and others.

There is a generalized interval system relevant for this piece. We use the simply transitive group action picture.

HINDEMITH: LUDUS TONALIS




28
pp
Cadence of
C#, F# again
33
mf
f
Tq of initial
38
pp
42
mf
47
p subito
cresc.


up to one note
* like K(GCD)
Retrograde of CFG
C F B♭
cresc.
mp
mf
cresc.
f
allargando e creseendo
[f]

Musical Example: Hindemith, *Ludus Tonalis*, Fugue in G

Our 24 element musical space S consists of the transposed and inverted forms of the *ordered* pitch-class segment $\langle G, C, D \rangle = \langle 7, 0, 2 \rangle$, which forms a motive in the subject in measure 2.

Transposed Forms	Inverted Forms
$\langle 7, 0, 2 \rangle$	$\langle 5, 0, 10 \rangle$
$\langle 8, 1, 3 \rangle$	$\langle 6, 1, 11 \rangle$
$\langle 9, 2, 4 \rangle$	$\langle 7, 2, 0 \rangle$
$\langle 10, 3, 5 \rangle$	$\langle 8, 3, 1 \rangle$
$\langle 11, 4, 6 \rangle$	$\langle 9, 4, 2 \rangle$
$\langle 0, 5, 7 \rangle$	$\langle 10, 5, 3 \rangle$
$\langle 1, 6, 8 \rangle$	$\langle 11, 6, 4 \rangle$
$\langle 2, 7, 9 \rangle$	$\langle 0, 7, 5 \rangle$
$\langle 3, 8, 10 \rangle$	$\langle 1, 8, 6 \rangle$
$\langle 4, 9, 11 \rangle$	$\langle 2, 9, 7 \rangle$
$\langle 5, 10, 0 \rangle$	$\langle 3, 10, 8 \rangle$
$\langle 6, 11, 1 \rangle$	$\langle 4, 11, 9 \rangle$

Musical Example: Hindemith, *Ludus Tonalis*, Fugue in G

The functions $T_n : \mathbb{Z}_{12} \to \mathbb{Z}_{12}$ and $I_n : \mathbb{Z}_{12} \to \mathbb{Z}_{12}$ induce functions on S by componentwise action. Recall that $T_1 : \mathbb{Z}_{12} \to \mathbb{Z}_{12}$ is a rotation of the twelve sided musical clock by 1/12th of a turn, and $I_0 : \mathbb{Z}_{12} \to \mathbb{Z}_{12}$ is a reflection about the 0-6 axis. In other words T_1 and I_0 generate the *group of symmetries* of the 12-gon, commonly known as the *dihedral group of order 24*. This group is $\{T_n | n \in \mathbb{Z}_{12}\} \cup \{I_n | n \in \mathbb{Z}_{12}\}$ and we call it the T/I group.

All of this amounts to a group action of the dihedral group of order 24 on the musical space S! Further, this group action is simply transitive, so we have a generalized interval system.

Musical Example: Hindemith, *Ludus Tonalis*, Fugue in G

Another important transformation (function) is the *contextual inversion* $J : S \to S$.

$J(x) :=$ that form of opposite type as x that has the same first two pitch classes but in the opposite order. Very audible!

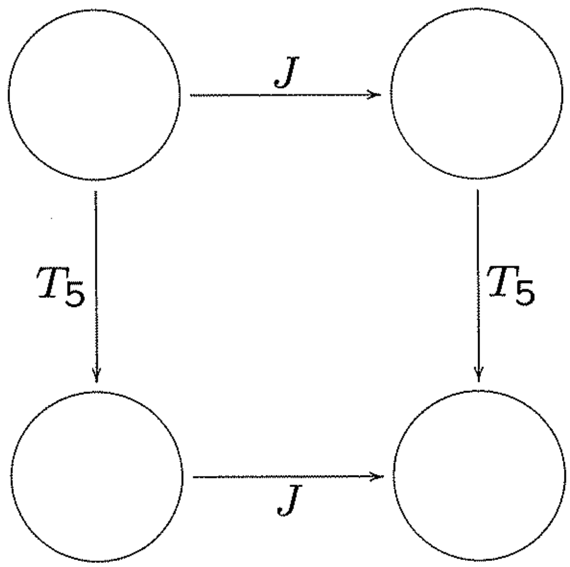
$J\langle 7, 0, 2\rangle = \langle 0, 7, 5\rangle$

This is contextual in that it is defined in terms of the aggregrate pcseg rather than the components, and it depends on whether the input is a transposed form or an inverted form.

J occurs in every instance of the subject, although one needs to use the underlying unordered pitch class set in the piece.

Musical Example: Hindemith, *Ludus Tonalis*, Fugue in G

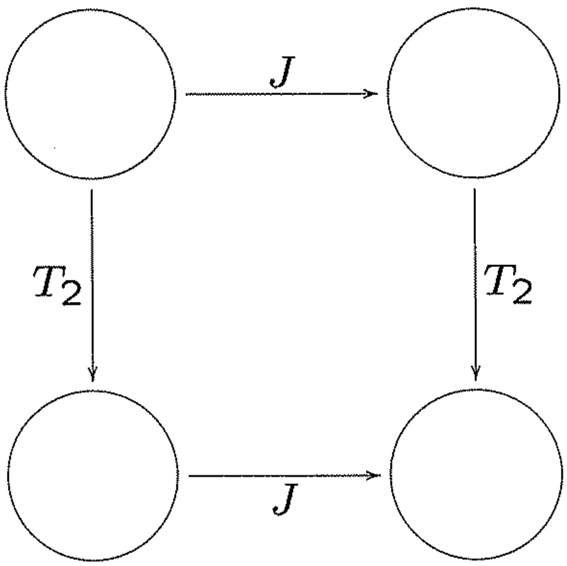
The diagram commutes!



It occurs in measures 2 and 4, 9 and 16, 37 and 39, 55 and 57.

Musical Example: Hindemith, *Ludus Tonalis*, Fugue in G

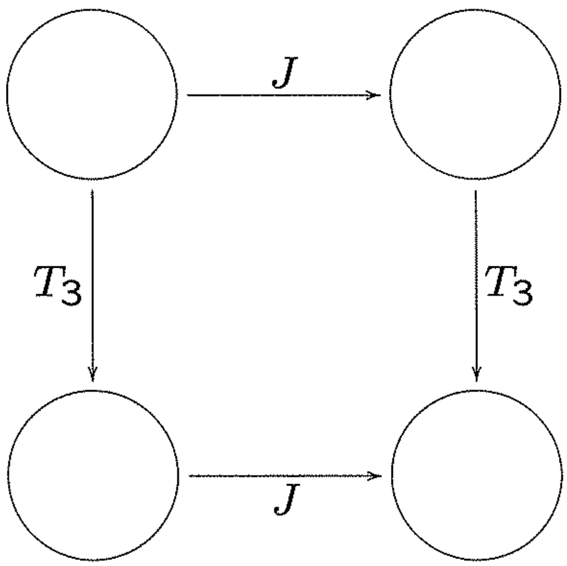
The diagram commutes!



It occurs in measures 25 and 26, 60 and 62, and 62 and 64.

Musical Example: Hindemith, *Ludus Tonalis*, Fugue in G

The diagram commutes!



It occurs in measures 26 and 27.

Note $T_3 \circ T_2 = T_5$. Note also T_2 and T_5 occur in the initial pcseg.

Musical Example: Hindemith, *Ludus Tonalis*, Fugue in G

In fact J commutes with all transpositions and inversions. In other words, J is an element of the centralizer of $T/I \subset Sym(S)$. This is because J is an *interval preserving function* for the GIS associated to the simply transitive group action of T/I on S.

Summary

Today we began our discussion of transformational theory. Transformational theory asks, which transformations are idiomatic for a piece? The idiomatic transformations in Hindemith, Ludus Tonalis, Fugue in G were the contextual inversion J and the transpositions T_2, T_3, and T_5. These fit together in interesting *networks* that allowed us to find a good way of hearing the piece and a good way of communicating that way of hearing.

We also introduced David Lewin's notion of generalized interval system, which abstracts the relevant features of intervals. In our musical example we viewed a generalized interval system as a simply transitive group action.