Mathematics and Music Lecture 4

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Transformational Theory

Topics:
Which transformations are idiomatic for a piece?
Neo-Riemannian Theory, Tonnetze, contextual inversions...

Mathematical Tools:
group actions on sets, topology

Today we'll talk about Neo-Riemannian Theory and wrap up our discussion of Transformational Theory.

Neo-Riemannian Theory

This branch of Transformational Theory began in

David Lewin. "A Formal Theory of Generalized Tonal Functions." *Journal of Music Theory* 26/1 (1982): 32-60.

with a transformational approach to triadic relations.

These transformations arose in the work of the 19th century music theorist Hugo Riemann, and have a pictorial description on the Oettingen/Riemann *Tonnetz*. Topology enters the picture here.

Neo-Riemannian Theory

Recent work focuses on the neo-Riemannian operations P, L, and R. They generate a dihedral group, called the *neo-Riemannian group.* As we'll see, this group is *dual* to the T/I group in the GIS sense of Lewin.

P, L, and R have several distinguishing features. They preserve two common tones, in much the same way that J preserves two common tones in our analysis of Hindemith, *Ludus Tonalis*, Fugue in G. But for P, L, and R, the third note moves by a minimal distance. This feature is known as *voice leading parsimony.*

Major and Minor Triads

The major and minor triads are all obtained by transposing and inverting the C major triad $\langle 0,4,7\rangle$. Let S denote the set of 24 major and minor triads, *i.e.* the set whose elements are the following.

Transposed Forms	Inverted Forms
$C = \langle 0,4,7\rangle$	$\langle 0,8,5\rangle = f$
$C\sharp = D\flat = \langle 1,5,8\rangle$	$\langle 1,9,6\rangle = f\sharp = g\flat$
$D = \langle 2,6,9\rangle$	$\langle 2,10,7\rangle = g$
$D\sharp = E\flat = \langle 3,7,10\rangle$	$\langle 3,11,8\rangle = g\sharp = a\flat$
$E = \langle 4,8,11\rangle$	$\langle 4,0,9\rangle = a$
$F = \langle 5,9,0\rangle$	$\langle 5,1,10\rangle = a\sharp = b\flat$
$F\sharp = G\flat = \langle 6,10,1\rangle$	$\langle 6,2,11\rangle = b$
$G = \langle 7,11,2\rangle$	$\langle 7,3,0\rangle = c$
$G\sharp = A\flat = \langle 8,0,3\rangle$	$\langle 8,4,1\rangle = c\sharp = d\flat$
$A = \langle 9,1,4\rangle$	$\langle 9,5,2\rangle = d$
$A\sharp = B\flat = \langle 10,2,5\rangle$	$\langle 10,6,3\rangle = d\sharp = e\flat$
$B = \langle 11,3,6\rangle$	$\langle 11,7,4\rangle = e$

Simply Transitive Group Action

The 24 element T/I group acts simply transitively on the set S of major and minor triads. In other words, S is the musical space of a generalized interval system.

The Neo-Riemannian Transformations

Recall the contextual inversion J in our analysis of Hindemith, *Ludus Tonalis*, Fugue in G. The inputs overlapped with the outputs in two pitch classes, for example $J\langle 7, 0, 2\rangle = \langle 0, 7, 5\rangle$. We define analogous functions on the major and minor triads.

- Let $P(x)$ be that form of opposite type as x with the first and third notes switched. For example

$$P\langle 0, 4, 7\rangle = \langle 7, 3, 0\rangle$$

$$P\langle 3, 11, 8\rangle = \langle 8, 0, 3\rangle.$$

The Neo-Riemannian Transformations

- Let $L(x)$ be that form of opposite type as x with the second and third notes switched. For example

$$L\langle 0,4,7\rangle = \langle 11,7,4\rangle$$

$$L\langle 3,11,8\rangle = \langle 4,8,11\rangle.$$

- Let $R(x)$ be that form of opposite type as x with the first and second notes switched. For example

$$R\langle 0,4,7\rangle = \langle 4,0,9\rangle$$

$$R\langle 3,11,8\rangle = \langle 11,3,6\rangle.$$

Musical Aspects of Neo-Riemannian Transformations

The neo-Riemannian transformations P, L, and R are highly musical.

- The inputs and outputs always overlap in two pitch classes. Audible!
- The third pitch class moves by a minimum amount. Audible! (Parsimonius Voice Leading)

Musical Aspects of Neo-Riemannian Transformations

- P=parallel
  $P(C \text{ major}) = c$ minor

  L=leading tone exchange
  $L(C \text{ major}) = e$ minor

  R=relative
  $R(C \text{ major}) = a$ minor

- $P, L,$ and R are contextual inversions: the axis of inversion depends on the input.

$$P\langle x_1, x_2, x_3\rangle = I_{x_1+x_3}\langle x_1, x_2, x_3\rangle$$

$$L\langle x_1, x_2, x_3\rangle = I_{x_2+x_3}\langle x_1, x_2, x_3\rangle$$

$$R\langle x_1, x_2, x_3\rangle = I_{x_1+x_2}\langle x_1, x_2, x_3\rangle.$$

Parsimonious Voice Leading

The third pitch class moves only by interval 1 in the case of P and L, and by interval 2 in the case of R.

$$P\langle 0,4,7\rangle = \langle 7,3,0\rangle$$

$$L\langle 0,4,7\rangle = \langle 11,7,4\rangle$$

$$R\langle 0,4,7\rangle = \langle 4,0,9\rangle$$

This is called *parsimonious voice leading*.

"parsimony=law of the shortest way"

J does not have this feature: $J\langle 7,0,2\rangle = \langle 0,7,5\rangle$. In fact the major and minor triads form the unique set class that supports parsimonious voice leading! (Cohn, 1997)

The Neo-Riemannian PLR Group

Definition 1 *The PLR group is the group whose set consists of all possible compositions of P,L, and R. The group operation is function composition.*

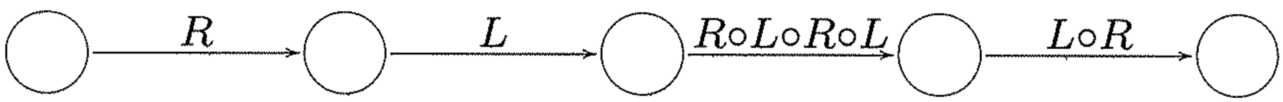
Theorem 1 *The PLR group is dihedral of order 24 and is generated by L and R.*

In other words, the PLR group is isomorphic to the group of symmetries of the 12-gon. Hence, isomorphic to the T/I group.

Theorem 2 *The PLR group acts simply transitively on the set of major and minor triads, and hence forms a generalized interval system.*

The PLR Group

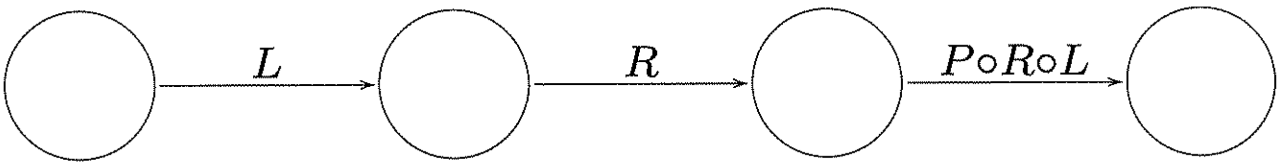
Example 1 *The Elvis Progression I-VI-IV-V-I from 50's Rock is as follows.*



The Elvis Progression can be found in "Stand by Me" for example.

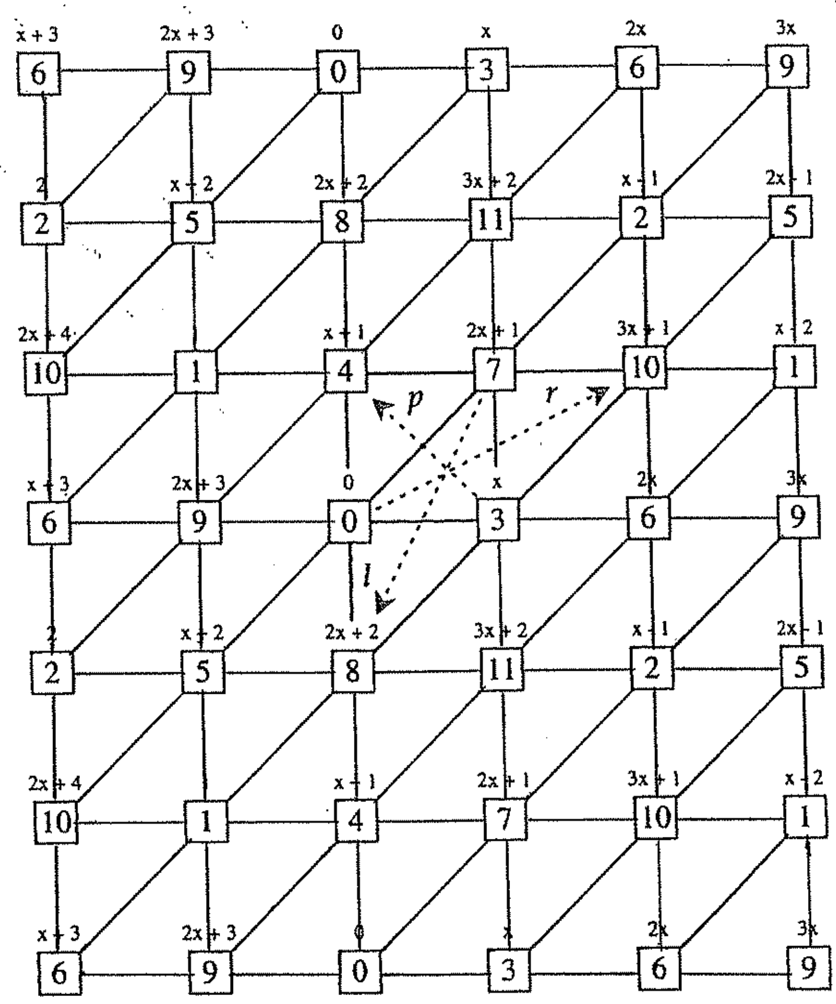
"Oh! Darling" from the Beatles

The progression $f\sharp$ minor, D major, b minor, and E major is obtained from the following application of the PLR group.



The Oettingen/Riemann *Tonnetz*

The functions P, L, and R have a beautiful description in terms of the Oettingen/Riemann *Tonnetz*.



Rick Cohn. "Neo-Riemannian Operations, Parsimonious Trichords, and Their Tonnetz Representations." Journal of Music Theory 41/1 (1997): 1–66.

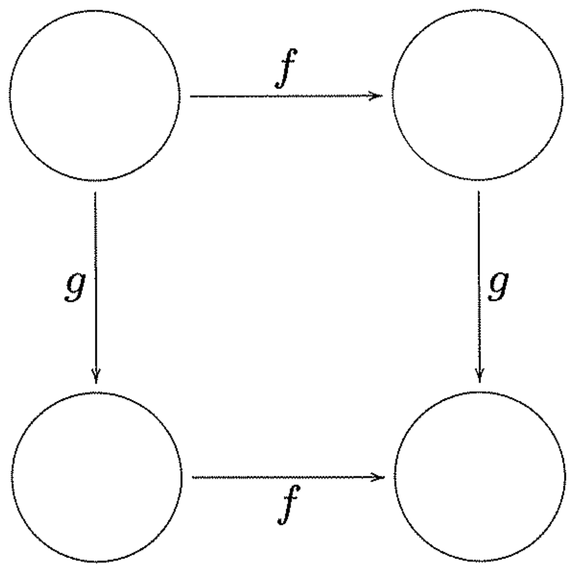
The Oettingen/Riemann *Tonnetz*

On the Oettingen/Riemann *Tonnetz*, a vertex is a pitch class. A major or minor chord is a triangle connecting three adjacent vertices. P, L and R correspond to flipping triangles about a their respective sides. The axis of reflection is the edge connecting the common tones preserved by P, L, or R respectively.

On the southwest to northeast diagonal we have the circle of fifths. On the horizontal axis we have the circle of minor thirds. On the vertical axis we have the circle of major thirds. Notice that this graph is periodic.

Dual Groups

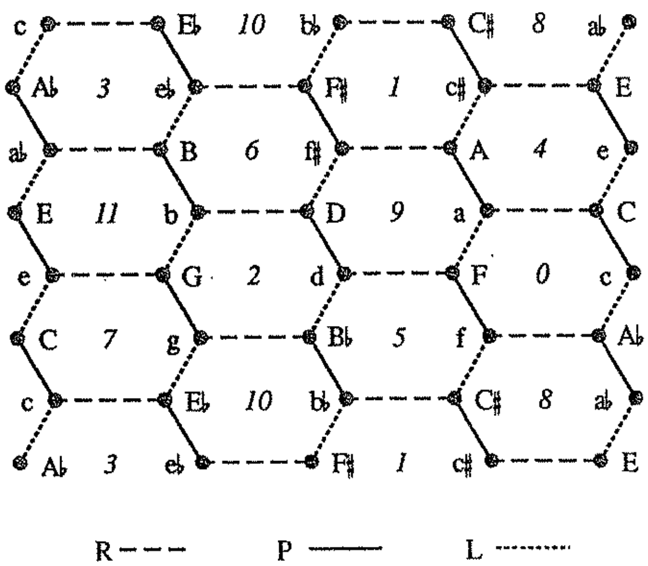
The T/I group is *dual* to the PLR group in the sense that the diagram



commutes for any f in the PLR group and any g in the T/I group. In fact, the T/I group and the PLR group give rise to *dual* GIS's. Ramon Satyendra and I have generalized this beyond the set S of major and minor chords.

Dual Graphs

But these groups aren't the only things that are dual! Let's consider the *dual graph* to the *Tonnetz* graph. This means: draw a vertex inside of every triangle, and connect two vertices whenever the corresponding triangles share an edge. The result is the graph below from Douthett and Steinbach. The vertices are now chords, and the edges are PLR operations. Note the periodicity!



Jack Douthett and Peter Steinbach. "Parsimonious Graphs: A Study in Parsimony, Contextual Transformations, and Modes of Limited Transposition." Journal of Music Theory 42/2 (1998): 241–263.

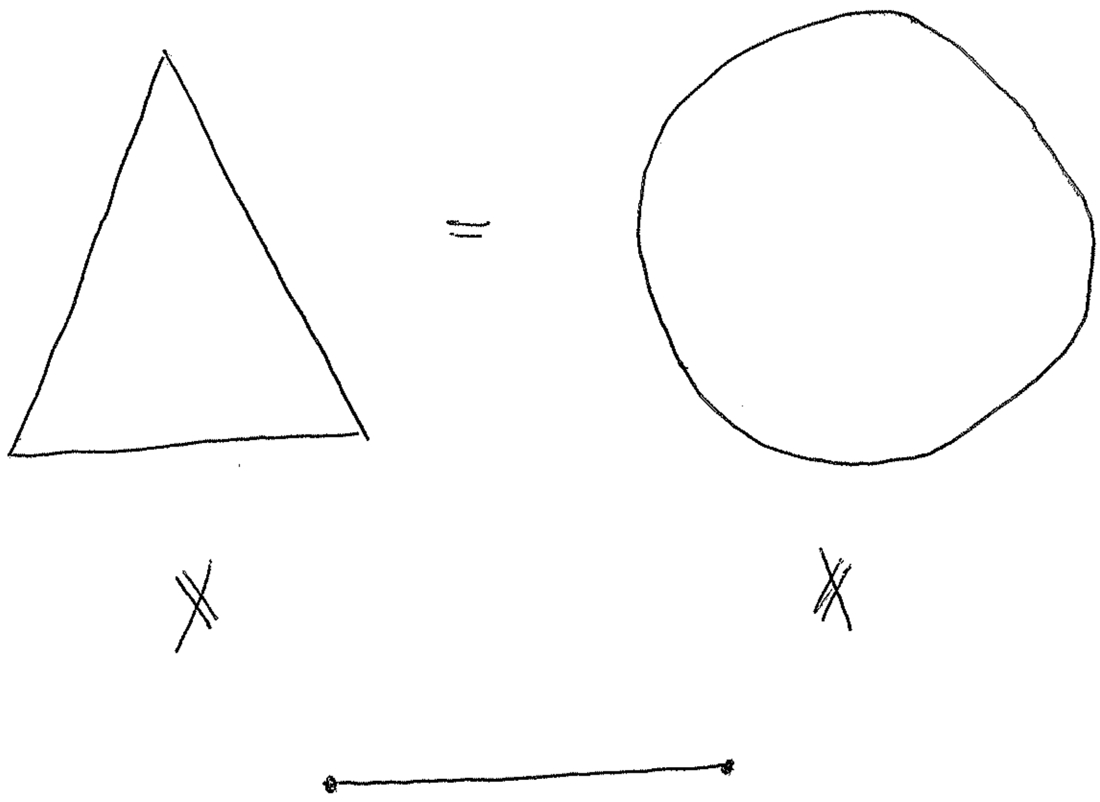
Topology and the Torus

Topology is a major branch of mathematics which studies qualitative questions about geometry. Two objects are qualitatively the same if one can be stretched, shrunk, or twisted into the other. Qualitative questions:

- Is the object connected?
- Does it have boundary?
- How many holes does it have?

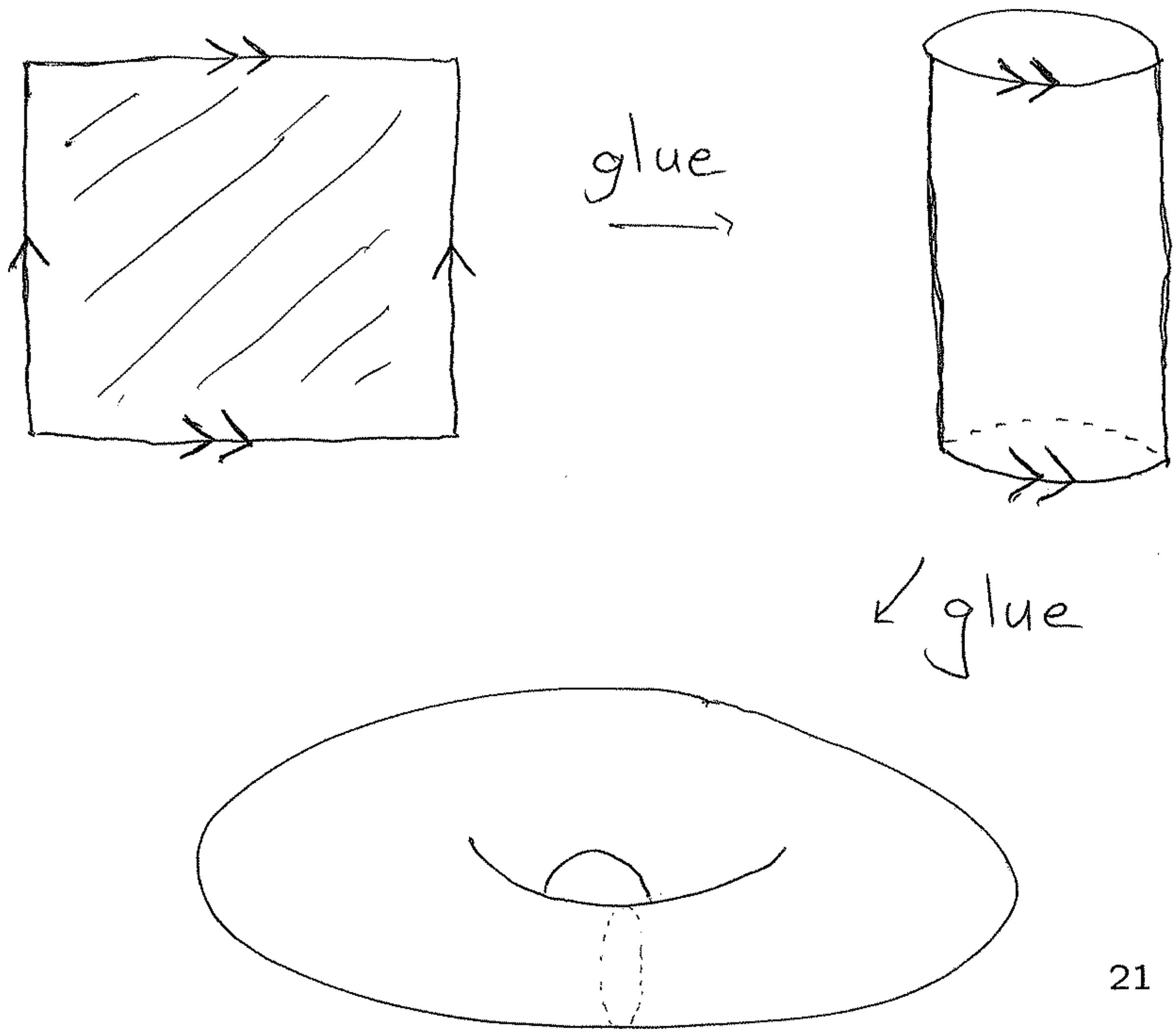
Topology and the Torus

The triangle and circle are the same qualitatively, but they are different from the line segment.



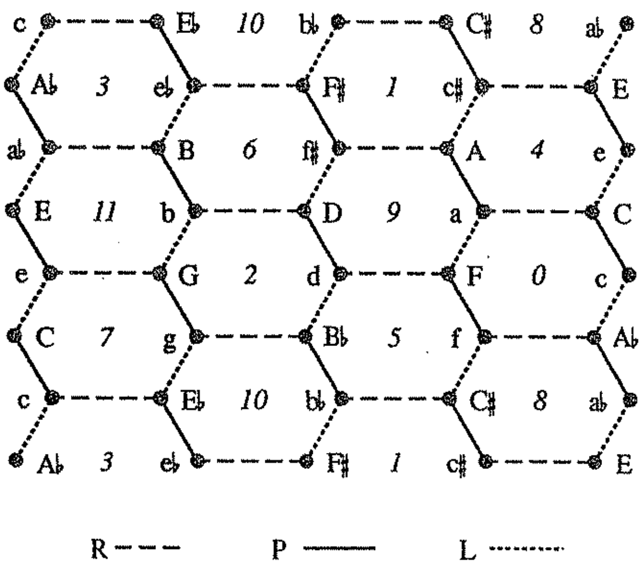
Topology and the Torus

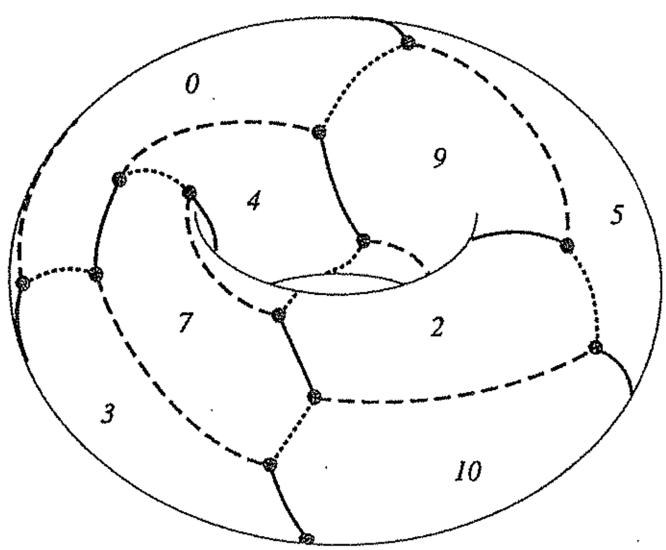
The torus is obtained from a square sheet by gluing the vertical edges and then the horizontal edges.



Topology and the Torus

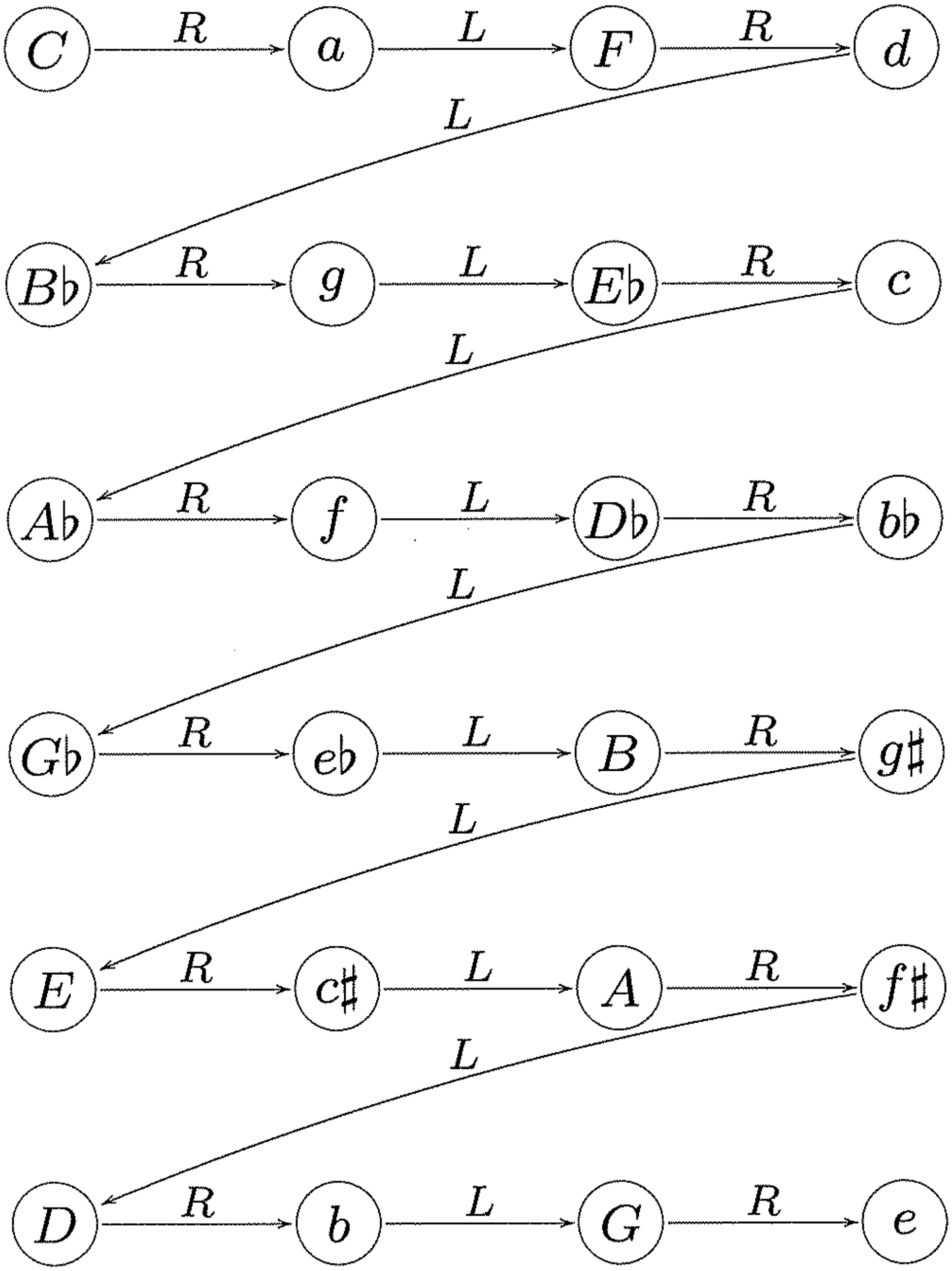
The *Tonnetz* graph and Douthett and Steinbach's PLR graph are actually graphs on the torus because of their periodicities!





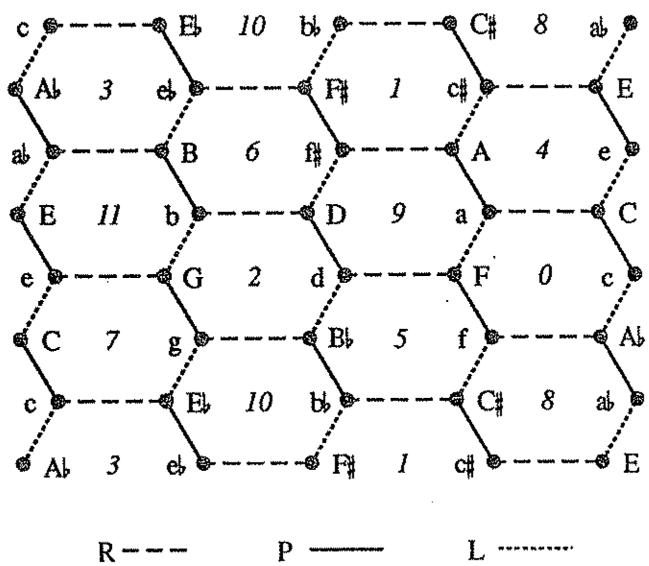
Beethoven's Ninth Symphony

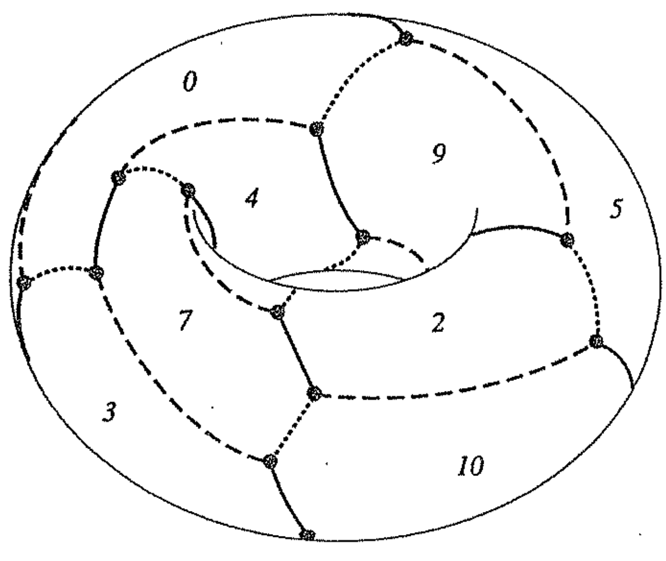
The remarkable chord progression in measures 143-176 of the second movement of Beethoven's Ninth Symphony is:



Beethoven's Ninth Symphony

This chord progression is precisely a path on Douthett and Steinbach's torus! It nearly covers the whole torus without repeats!





Summary

- In this lecture series I have introduced some of the conceptual categories that music theorists use to make aural impressions into vivacious ideas in the sense of Hume.

- The conceptual categories of interest to mathematicians are: arithmetic modulo 12, equivalence relations, transposition and inversion, properties and classification of scales, naive set theory, the hexachord theorem, generalized interval systems, simply transitive group actions, the *Tonnetz*, the PLR group, its associated graph, duality, and the torus.

Summary

- We have used these tools to find good ways of hearing music from Schumann, Bartok, Debussy, Schoenberg, Wagner, Hindemith, the Beatles, and Beethoven. Many of these analyses would have been impossible without mathematics.

- I hope this introduction to Scale Theory, Set Theory, and Transformational Theory turned your impressions of music theory into tangible ideas!