Additions to Double Categories and Pseudo Algebras

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Towards Strict 2-Algebras and Crossed Modules

We continue our comparison of strict 2-algebras over the 2-theory of categories with crossed modules. We first state the update of Theorem 6 from the first part of the slides (the notion of group action there wasn't so good, so we avoid it here). Then we define the notions in the 2-equivalence.

Theorem 1 *(F. 2007) The following 2-categories are 2-equivalent.*
(i) Crossed modules under groups I.
(ii) 2-Groups $\mathcal{C}$ equipped with a 2-functor $I \longrightarrow \mathcal{C}$.
(iii) Double groups with folding and vertical group I.
(iv) Strict 2-algebras over the 2-theory of categories with everything invertible and one object, with underlying I.

Recall the following slide.

Towards Strict 2-Algebras and Crossed Modules

It is often useful to consider one-object cases of categorical concepts and to recognize them as something familiar.

groupoids	$\subseteq$ 2-groupoids	$\subseteq$ double groupoids
groups	$\subseteq$ 2-groups	$\subseteq$ double groups

Theorem 2 *(Verdier, Brown-Spencer 1976,...) 2-groups are equivalent to crossed modules.*

Theorem 3 *(Brown-Spencer 1976) Edge symmetric double groups with folding structure with trivial holonomy are equivalent to crossed modules.*

Question: What is a one-object strict 2-algebra over the 2-theory of categories with everything iso?

Crossed Modules

Definition 1 *A crossed module is a group homomorphism* $\partial : H \longrightarrow G$ *with a left action*

$$G \times H \longrightarrow H$$

$$(g, \alpha) \mapsto {}^{g}\alpha$$

by automorphisms such that:

1. $\partial({}^{g}\alpha) = g\partial(\alpha)g^{-1}$ *for all* $\alpha \in H$ *and* $g \in G$*.*

2. ${}^{\partial(\alpha)}\beta = \alpha\beta\alpha^{-1}$ *for all* $\alpha, \beta \in H$*.*

Example 1 *Any group* I *can be considered a crossed module with* $\{e\} \hookrightarrow I$*.*

Crossed Modules

Example 2 *The inclusion of a normal subgroup* $H \lhd G$ *is a crossed module.*

Example 3 *Let* $(X, A, *)$ *be a based pair of topological spaces. Then*

$$\partial : \pi_2(X, A, *) \longrightarrow \pi_1(A, *)$$

is a crossed module.

Theorem 4 *(Mac Lane-Whitehead 1950) Crossed modules model path-connected homotopy 2-types via this example.*

Crossed Modules Under Groups

Definition 2 *A crossed module under a group consists of a group I, a crossed module $\partial : H \longrightarrow G$, and a morphism of crossed modules*

$$\begin{array}{ccc} \{e\} & \longrightarrow & I \\ \downarrow & & \downarrow \\ H & \xrightarrow[\partial]{} & G. \end{array}$$

Strict 2-Algebras and Crossed Modules

Recall the Theorem.

Theorem 5 *(F. 2007) The following 2-categories are 2-equivalent.*
(i) Crossed modules under groups I.
(ii) 2-Groups $\mathcal{C}$ equipped with a 2-functor $I \longrightarrow \mathcal{C}$.
(iii) Double groups with folding and vertical group I.
(iv) Strict 2-algebras over the 2-theory of categories with everything invertible and one object, with underlying I.

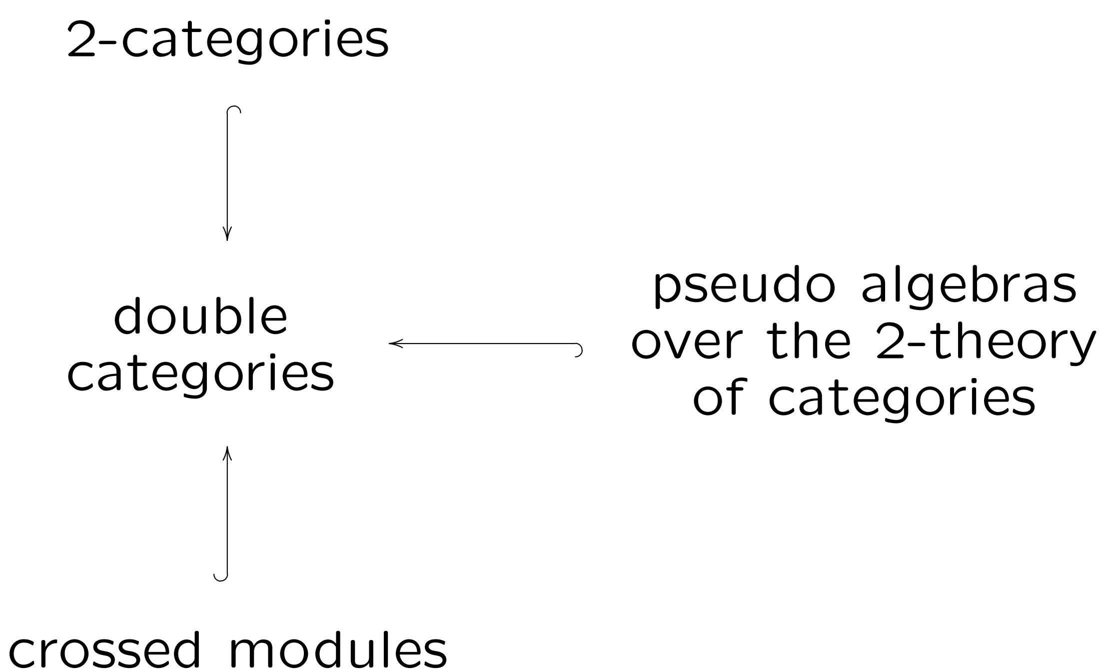
In the case of a trivial I, this says 2-groups are equivalent to crossed modules. We see that the Theorem of Brown and Spencer mentioned above is a special case.

Thus we have a comparison of crossed modules under groups with one-object strict 2-algebras over the 2-theory of categories with everything invertible.

The purpose of this work was to build a bridge from (pseudo) algebras over the 2-theory of categories to traditional categorical concepts. The strict 2-algebras are 2-equivalent to double categories with folding.

This is somewhat a surprise: these strict 2-algebras are neither enriched 2-categories, nor internal categories in Cat, but instead an intermediate notion. Thus, categorification via pseudo algebras over 2-theories is different than enrichment and internalization.

Conclusion


2-categories
double categories
pseudo algebras over the 2-theory of categories
crossed modules