Double Categories and Pseudo Algebras

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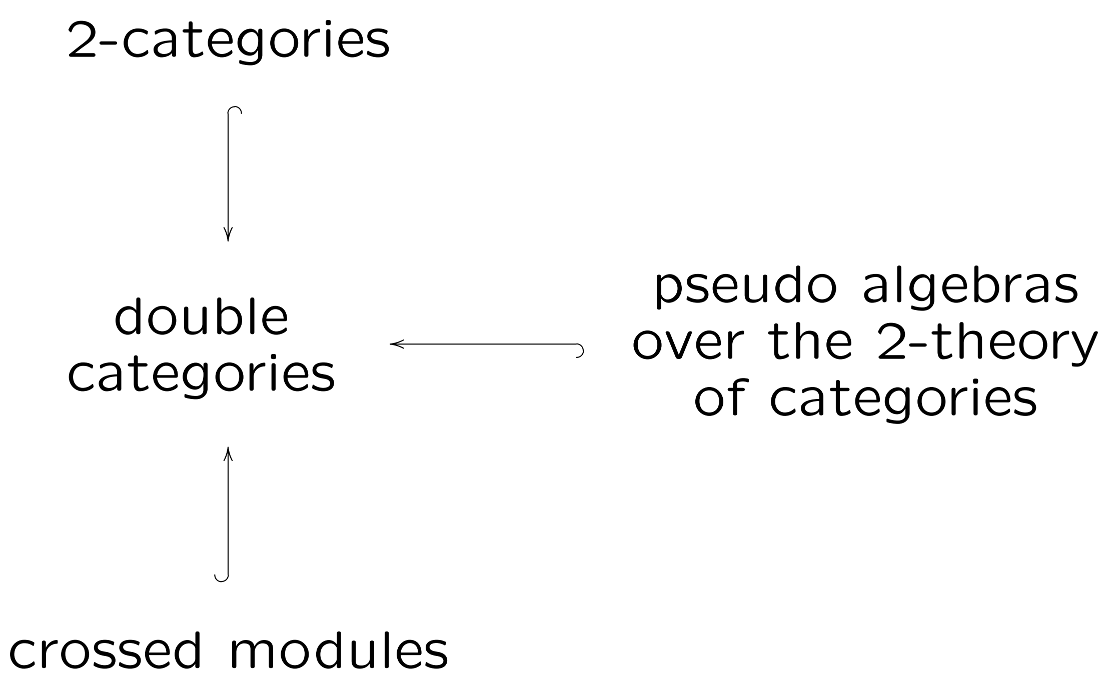
Overview


2-categories
double categories
pseudo algebras over the 2-theory of categories
crossed modules

Chronology

1942-1945 Eilenberg-Mac Lane: category theory

1946, 1950 Whitehead-Mac Lane: crossed modules, homotopy 2-types

1963 Ehresmann: double categories

1970's R. Brown: 2-groups, crossed modules, Van Kampen Theorems

1988 Segal Bourbaki talk: a CFT "is" a cocycle for elliptic cohomology

1991 Mac Lane: coherence in CFT

2002-2005 Fiore, Hu, Kriz: pseudo algebras over theories and 2-theories as a rigorous foundation of CFT

2-Categories

Definition 1 *A 2-category* $\mathbf{C}$ *is a category enriched in categories, i.e.*

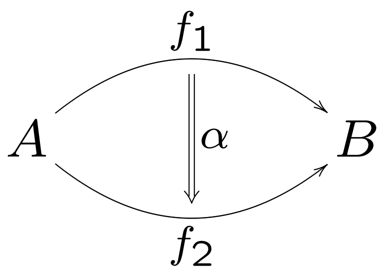
- *a set of objects* $Obj\,\mathbf{C}$
- *for each object* A *and* B *a category* $Mor_{\mathbf{C}}(A,B)$
- *composition functors*

  $$Mor_{\mathbf{C}}(B,C) \times Mor_{\mathbf{C}}(A,B) \xrightarrow{\circ} Mor_{\mathbf{C}}(A,C)$$

- *identities* $1_A \in Mor_{\mathbf{C}}(A,A)$

which satisfy the usual axioms for a category.

Examples



Example 1 *Any category is a 2-category with discrete morphism categories.*

Example 2 *Topological spaces, continuous maps, homotopy classes of homotopies.*

Example 3 *Categories, functors, and natural transformations form the 2-category* Cat.

Example 4 *Rings, bimodules, bimodule maps form a bicategory.*

Double Categories

Definition 2 *(Ehresmann 1963) A double category $\mathbb{D}$ is an internal category in $\mathcal{C}at$.*

Definition 3 *A double category $\mathbb{D}$ consists of*
a set of objects,
a set of horizontal morphisms,
a set of vertical morphisms, and
a class of squares with source and target as follows

$$A \xrightarrow{f} B \qquad \begin{array}{c} A \\ {\scriptstyle j}\downarrow \\ C \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle j}\downarrow & \alpha & \downarrow{\scriptstyle k} \\ C & \xrightarrow[g]{} & D \end{array}$$

and compositions and units that satisfy axioms.

Compositions and Units for Morphisms in a Double Category

Horizontal:

$$A \xrightarrow{f_1} B \xrightarrow{f_2} C = \begin{bmatrix} f_1 & f_2 \end{bmatrix} = f_2 \circ f_1$$

$$A \xrightarrow{1^h_A} A \xrightarrow{f_1} B = f_1 = A \xrightarrow{f_1} B \xrightarrow{1^h_B} B$$

Vertical:

$$\begin{array}{c} A \\ \downarrow{\scriptstyle j_1} \\ B \\ \downarrow{\scriptstyle j_2} \\ C \end{array} = \begin{bmatrix} j_1 \\ j_2 \end{bmatrix} = j_2 \circ j_1$$

$$\begin{array}{c} A \\ \downarrow{\scriptstyle 1^v_A} \\ A \\ \downarrow{\scriptstyle j_1} \\ B \end{array} = \begin{array}{c} A \\ \downarrow{\scriptstyle j_1} \\ B \end{array} = \begin{array}{c} A \\ \downarrow{\scriptstyle j_1} \\ B \\ \downarrow{\scriptstyle 1^v_B} \\ B \end{array}$$

Compositions for Squares in a Double Category

Horizontal:

$$\begin{array}{ccccc} A & \xrightarrow{f_1} & B & \xrightarrow{f_2} & C \\ {\scriptstyle j}\downarrow & \alpha & {\scriptstyle k}\downarrow & \beta & \downarrow{\scriptstyle \ell} \\ D & \xrightarrow[g_1]{} & E & \xrightarrow[g_2]{} & F \end{array} \quad = \quad \begin{array}{ccc} A & \xrightarrow{[f_1 f_2]} & C \\ {\scriptstyle j}\downarrow & [\alpha\beta] & \downarrow{\scriptstyle \ell} \\ D & \xrightarrow[{[g_1 g_2]}]{} & F \end{array}$$

Vertical:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle j_1}\downarrow & \alpha & \downarrow{\scriptstyle k_1} \\ C & \xrightarrow{g} & D \\ {\scriptstyle j_2}\downarrow & \beta & \downarrow{\scriptstyle k_2} \\ E & \xrightarrow[h]{} & F \end{array} \quad = \quad \begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle \left[\begin{smallmatrix} j_1 \\ j_2 \end{smallmatrix}\right]}\downarrow & \left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right] & \downarrow{\scriptstyle \left[\begin{smallmatrix} k_1 \\ k_2 \end{smallmatrix}\right]} \\ E & \xrightarrow[h]{} & F \end{array}$$

Units for Squares in a Double Category

Horizontal:

$$\begin{array}{ccccc} A & \xrightarrow{1^h_A} & A & \xrightarrow{f} & B \\ {\scriptstyle j}\downarrow & i^h_j & {\scriptstyle j}\downarrow & \alpha & \downarrow{\scriptstyle k} \\ C & \xrightarrow[1^h_C]{} & C & \xrightarrow[g]{} & D \end{array} \quad = \quad \begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle j}\downarrow & \alpha & \downarrow{\scriptstyle k} \\ C & \xrightarrow[g]{} & D \end{array}$$

$$= \quad \begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{1^h_B} & B \\ {\scriptstyle j}\downarrow & \alpha & {\scriptstyle k}\downarrow & i^h_k & \downarrow{\scriptstyle k} \\ C & \xrightarrow[g]{} & D & \xrightarrow[1^h_D]{} & D \end{array}$$

Vertical:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle 1^v_A}\downarrow & i^v_f & \downarrow{\scriptstyle 1^v_B} \\ A & \xrightarrow{f} & B \\ {\scriptstyle j}\downarrow & \alpha & \downarrow{\scriptstyle k} \\ C & \xrightarrow[g]{} & D \end{array} \quad = \quad \begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle j}\downarrow & \alpha & \downarrow{\scriptstyle k} \\ C & \xrightarrow[g]{} & D \end{array} \quad = \quad \begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle j}\downarrow & \alpha & \downarrow{\scriptstyle k} \\ C & \xrightarrow{g} & D \\ {\scriptstyle 1^v_C}\downarrow & i^v_g & \downarrow{\scriptstyle 1^v_D} \\ C & \xrightarrow[g]{} & D \end{array}$$

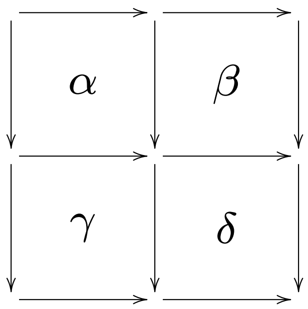
Axioms for a Double Category

All compositions are *associative* and *unital* (as above) and

$$\begin{bmatrix} i^h_{j_1} \\ i^h_{j_2} \end{bmatrix} = i^h_{\left[\begin{smallmatrix} j_1 \\ j_2 \end{smallmatrix}\right]}$$

$$\begin{bmatrix} i^v_{f_1} & i^v_{f_2} \end{bmatrix} = i^v_{[f_1 f_2]}.$$

Interchange Law:

If 

, then $\begin{bmatrix} \begin{bmatrix} \alpha & \beta \end{bmatrix} \\ \begin{bmatrix} \gamma & \delta \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \alpha \\ \gamma \end{bmatrix} & \begin{bmatrix} \beta \\ \delta \end{bmatrix} \end{bmatrix}$ and

we write $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$.

Examples of Double Categories

Let I be a 1-category.

$\Box I$:= double category of commutative squares in I

$Obj\,\Box I := Obj\,I$
$Hor\,\Box I := Mor\,I$
$Ver\,\Box I := Mor\,I$
$Sq\,\Box I :=$ commutative squares in I

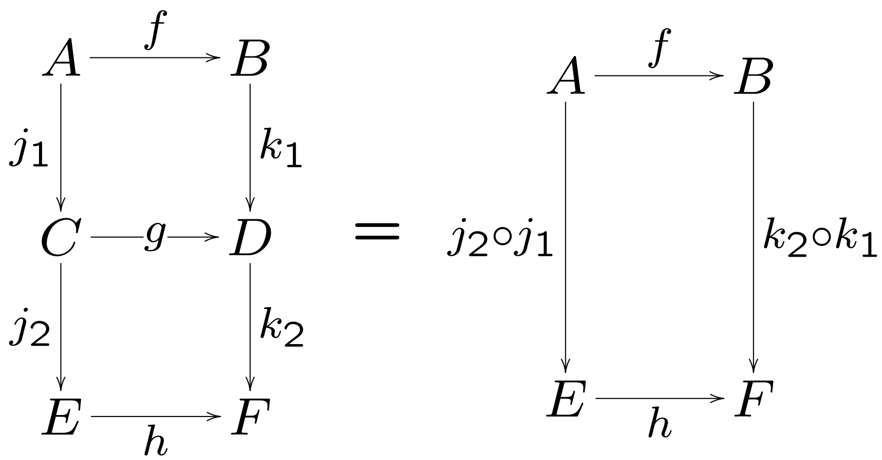
$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle j_1}\downarrow & & \downarrow{\scriptstyle k_1} \\ C & \xrightarrow{g} & D \\ {\scriptstyle j_2}\downarrow & & \downarrow{\scriptstyle k_2} \\ E & \xrightarrow[h]{} & F \end{array} \quad = \quad \begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle j_2\circ j_1}\downarrow & & \downarrow{\scriptstyle k_2\circ k_1} \\ E & \xrightarrow[h]{} & F \end{array}$$

Examples of Double Categories

Let I be a 1-category.

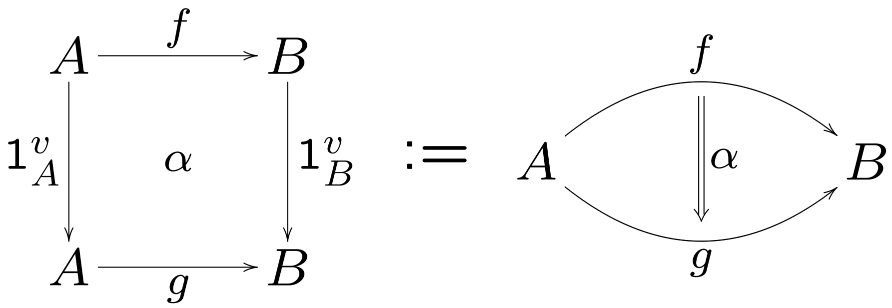
$\boxdot I$:= double category of not necessarily commutative squares in I

$Obj \boxdot I := Obj\, I$
$Hor \boxdot I := Mor\, I$
$Ver \boxdot I := Mor\, I$
$Sq \boxdot I$:= not necessarily commutative squares in I

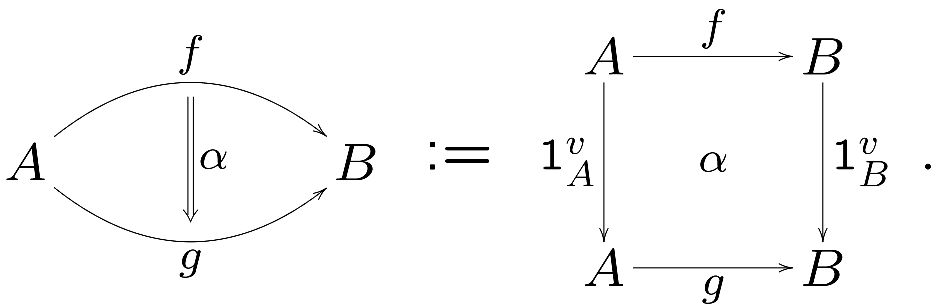


Examples of Double Categories

Every 2-category $\mathbf{C}$ is a double category with trivial vertical morphisms.



Definition 4 *The horizontal 2-category* $\mathbf{H}\mathbb{D}$ *of a double category* $\mathbb{D}$ *has objects* $Obj\,\mathbb{D}$*, morphisms* $Hor\,\mathbb{D}$*, and 2-cells*



Examples of Double Categories

Let $\mathbf{C}$ be a 2-category.

$\mathbb{Q}\mathbf{C}$:= Ehresmann's double category of *quintets* in $\mathbf{C}$ (1963)

$Obj\,\mathbb{Q}\mathbf{C} := Obj\,\mathbf{C}$
$Hor\,\mathbb{Q}\mathbf{C} := Mor\,\mathbf{C}$
$Ver\,\mathbb{Q}\mathbf{C} := Mor\,\mathbf{C}$

$$Sq\,\mathbb{Q}\mathbf{C} := \left\{ \begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle j}\downarrow & \alpha & \downarrow{\scriptstyle k} \\ C & \xrightarrow[g]{} & D \end{array} \;\middle|\; \alpha : k \circ f \Rightarrow g \circ j : A \to D \right\}$$

Theorem 1 *(Grandis-Paré 2004) The functor* $\mathbb{Q} : 2\text{-}Cat \longrightarrow Dbl$ *admits a right adjoint.*

Examples of Double Categories

$\mathbb{R}$ng:= pseudo double category or rings, bimodules, and equivariant maps

$Obj\,\mathbb{R}$ng := rings with identity
$Hor\,\mathbb{R}$ng := bimodules
$Ver\,\mathbb{R}$ng := homomorphisms of rings
$Sq\,\mathbb{R}$ng :=

$$\left\{ \begin{array}{c|l} \begin{array}{ccc} R & \xrightarrow{M} & S \\ {\scriptstyle j}\downarrow & \alpha & \downarrow{\scriptstyle k} \\ T & \xrightarrow[N]{} & U \end{array} & \begin{array}{l} \alpha : M \longrightarrow N \text{ group homomorphism} \\ \alpha(smr) = k(s)\alpha(m)j(r) \end{array} \end{array} \right\}$$

Examples of Double Categories

Let C be a topological category, *i.e.* $Obj\, C$ and $Mor\, C$ are topological spaces.

$\mathbb{P}'C$:= double category of Moore paths on C.

$Obj\, \mathbb{P}'C := Obj\, C$
$Hor\, \mathbb{P}'C := Mor\, C$
$Ver\, \mathbb{P}'C := P'(Obj\, C)$ = Moore paths in $Obj\, C$
$Sq\, \mathbb{P}'C := P'(Mor\, C)$ = Moore paths in $Mor\, C$

$P'X := \{(w, s) : s \geq 0, w : [0, s] \longrightarrow X\}$

Examples of Double Categories

A *worldsheet* is a real, compact, not necessarily connected, two dimensional, smooth manifold with complex structure and real analytically parametrized boundary components.

$\mathbb{W}$:= pseudo double category of worldsheets

$Obj\,\mathbb{W}$:= finite sets
$Hor\,\mathbb{W}(A,B)$:= worldsheets with inbound components labelled by A and outbound components by B
$Ver\,\mathbb{W}$:= bijections of finite sets
$Sq\,\mathbb{W}$:=

$$\left\{ \begin{array}{ccc} A & \xrightarrow{x} & B \\ {\scriptstyle j}\downarrow & \alpha & \downarrow{\scriptstyle k} \\ C & \xrightarrow[y]{} & D \end{array} \;\middle|\; \begin{array}{l} \alpha : x \longrightarrow y \text{ holomorphic diffeo.} \\ \alpha \text{ compatible with } j \text{ and } k \\ \alpha \text{ preserves boundary params.} \end{array} \right\}$$

Folding Structures

We introduce folding structures to compare algebras over the 2-theory of categories with double categories.

Definition 5 *A holonomy on a double category $\mathbb{D}$ is a 2-functor*

$$(\mathbf{V}\mathbb{D})_0 \longrightarrow \mathbf{H}\mathbb{D}$$

$$A \longmapsto \overline{A} = A$$

$$\begin{array}{c} A \\ \big\downarrow j \\ B \end{array} \longmapsto A \xrightarrow{\overline{j}} B$$

Example 5 *For a topological category C, a holonomy*

$$(\mathbf{V}\mathbb{P}'C)_0 \longrightarrow \mathbf{H}\mathbb{P}'C$$

assigns to a path of objects a morphism from the initial point to the terminal point, like in differential geometry.

Folding Structures

Definition 6 *A folding structure on a double category $\mathbb{D}$ consists of a holonomy $j \mapsto \overline{j}$ and bijections*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle j}\downarrow & \alpha & \downarrow{\scriptstyle k} \\ C & \xrightarrow[g]{} & D \end{array} \quad \overset{\Lambda}{\longleftrightarrow} \quad \begin{array}{ccc} A & \xrightarrow{[f\overline{k}]} & D \\ {\scriptstyle 1^v_A}\downarrow & \Lambda(\alpha) & \downarrow{\scriptstyle 1^v_D} \\ A & \xrightarrow[{[\overline{j}g]}]{} & D \end{array}$$

compatible with compositions and units.

A folding structure *horizontalizes* a double category.

Examples of Folding Structures

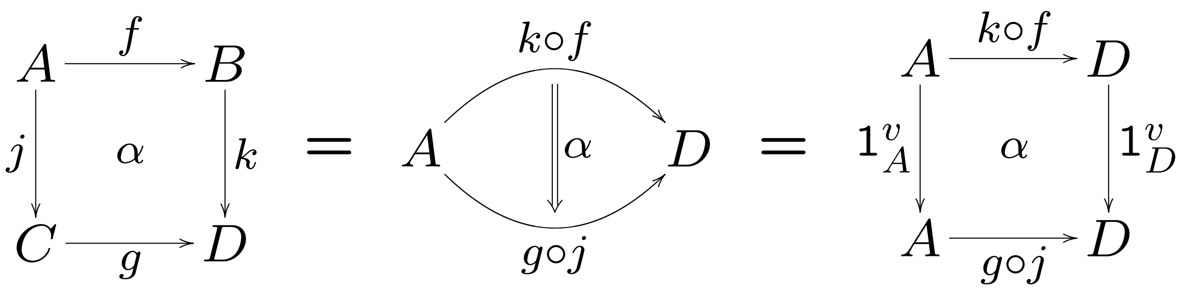
Let I be a 1-category.
$\Box I$=double category of commutative squares in I
$\boxdot I$=double category of not necessarily commutative squares in I

Then $\Box I$ and $\boxdot I$ each admit a unique folding structure.

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle k} \\ C & \xrightarrow[g]{} & D \end{array} \quad \overset{\Lambda}{\longleftrightarrow} \quad \begin{array}{ccc} A & \xrightarrow{k\circ f} & D \\ {\scriptstyle 1^v_A}\downarrow & & \downarrow{\scriptstyle 1^v_D} \\ A & \xrightarrow[g\circ j]{} & D \end{array}$$

Examples of Folding Structures

Let $\mathbf{C}$ be a 2-category. Then $\mathbb{Q}\mathbf{C}$ admits a folding structure by definition.



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Holonomy:

$$\overline{j} := T_j = \text{ the } (T, R)\text{-module } T$$

Folding:

$$\Lambda(\alpha) : U_k \bigotimes_S M \Longrightarrow N \bigotimes_T T_j$$

$$u \otimes m \longmapsto (u \cdot \alpha(m)) \otimes 1_T$$

Examples of Folding Structures

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Holonomy:

bijection $\longmapsto$ labelled union of infinitely thin annuli

Folding:

relabel x and y

Comparison Theorems

Theorem 2 *(Brown-Mosa 1999, F. 2006) The notions of folding structure and connection pair are equivalent.*

Theorem 3 *(F. 2006) The 2-category of strict 2-algebras over the 2-theory of categories is 2-equivalent to the 2-category of double categories with folding structures and invertible vertical morphisms.*

The pseudo version of the theorem also holds.