INTRODUCTION TO PSEUDO ALGEBRAIC STRUCTURES IN CONFORMAL FIELD THEORY

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Conformal field theory was described mathematically by Graeme Segal in the late 80's. Part of his description is a functor from a manifold category to a Hilbert space category, although various coherence details of this assignment were not written down. On the other hand we have elliptic cohomology. It was defined by Hopkins and Miller in terms of homotopy theory. A major problem in algebraic topology is to find a geometric definition of elliptic cohomology in the same sense that K-theory and de Rham theory are geometrically defined.

Segal suggested a geometric definition of elliptic cohomology in terms of conformal field theory in his 1988 Bourbaki talk. The inspiration for this idea was an analogy with K-theory. Since then, many people have worked on this problem. The enriched elliptic objects of Stolz and Teichner, the 2-vector bundles of Baas, Dundas, and Rogness, as well as the pseudo algebras over 2-theories of Hu and Kriz are a few examples.

The work described in this talk is a contribution to the categorical foundations of the approach formulated by Hu and Kriz. The main results in the talk show that certain weak 2-categorical limits, colimits, and adjoints exist in the context of pseudo algebras over theories and 2-theories. Many of these results can be found in some form in the literature. The existence of these bicategorical limits allows one to speak of stacks of pseudo algebras in this definition. A stack is a contravariant pseudofunctor from a Grothendieck site to a 2-category which takes Grothendieck covers to bilimits. The target category, usually pseudo algebras in this talk, should therefore admit bilimits or even pseudolimits. This is the reason these notions of limits were investigated in the 2-category of pseudo algebras.

The general philosophy is that of categorification: sets are replaced by categories, maps by functors, equalities by coherence isos, and relations by coherence diagrams. Lawvere theories are particularly convenient for this process because they provide a well defined method for writing down coherence isomorphisms and coherence diagrams for most algebraic structures on a set: for each operation of theories one obtains a coherence iso and for each relation of theories one obtains a coherence diagram. This leads to the concept of a pseudo algebra over a theory, *e.g.* the category of small sets with a fixed choice of disjoint union form a pseudo commutative monoid.

Before moving to the pseudo case, the talk introduces theories and strict algebras over them. If X is a set, then the endomorphism theory $End(X)$ is an example of a theory: it is a category whose objects are $0, 1, 2$ and whose morphisms from m to n are set maps $X^m \to X^n$. One can see the defining property for theories in this example: the object n is the product of n copies of 1. A strict algebra (or model) X for a theory T is a set X equipped with a morphism of theories $T \to End(X)$.

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This definition is very similar to the definition of a representation of a Lie algebra L on a vector space V as a Lie algebra homomorphism $L \to End(V)$.

The 2-category of pseudo algebras over a theory T admits all weighted pseudo limits and weighted bicolimits. This can be shown directly. There is also a 2-monadic description of the 2-category of pseudo T-algebras, which proves the result for pseudolimits of strict 2-functors as a consequence of the paper from Blackwell, Kelly, and Power. The 2-category of pseudo algebras over a 2-theory also admits pseudo limits.

The notions of 2-theory and pseudo algebra over a 2-theory are defined by analogy to the theory situation. Instead of a set or category X, one now has a strict 2-functor $X : I^k \to Cat$ and the associated 2-theory $End(X)$. An algebra over a 2-theory is then a morphism of 2-theories into the endomorphism 2-theory. The coherence isos and coherence diagrams for pseudo algebras over 2-theories are described in terms of operations and relations, just like for pseudo algebras over theories.

Pseudo algebras over 2-theories were developed to capture the algebraic structure given by disjoint union and gluing on the category of rigged surfaces. A rigged surface is a real, compact, not necessarily connected two dimensional manifold with complex structure and analytically parametrized boundary components. Boundary components are called inbound or outbound depending on the direction of the parametrization compared to the orientation inherited from the complex structure. We say that the manifold is labelled if the collection of inbound components and the collection of outbound components are each equipped with bijections to two (not necessarily disjoint) finite sets. Naively, the operation of gluing will identify an inbound component and an outbound component with the same label according to the rule $f(z) = g(z)$ for parametrizations f and g of the respective boundary components.

The difficulty is that the operations of disjoint union and gluing are indexed by the labellings as well. Disjoint union and gluing are not strictly associative or distributive, so coherence ismorphisms must also be part of the data. These coherences are neatly encoded in the description of the rigged surfaces as a pseudo algebra over the 2-theory of commutative monoids with cancellation.

Finally, a CFT is defined as a morphism of stacks which takes the stack of rigged surfaces to a stack on a Hilbert space. This captures Segal's idea of assigning Hilbert space data to manifold data in a coherent fashion. The stack of rigged surfaces is described in the paper of Hu and Kriz on elliptic cohomology and in the paper on the website mentioned in the talk .

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