Pseudo Algebraic Structures in CFT

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Introduction

This work is a contribution to the categorical foundations of CFT.

A CFT is a morphism of stacks of pseudo commutative monoids with cancellation. (Hu, Kriz)

Bicategorical versions of limits, colimits, and adjoints are essential to this approach.

General Philosophy

- equalities are replaced by coherence isos
- these coherence isos satisfy coherence diagrams
- bijection of sets is replaced by equivalence of categories

Examples

A *pseudo functor* between 2-categories preserves compositions and identities up to coherence isos, which satisfy coherence diagrams.

A *pseudo natural transformation* $\alpha : F \to G$ between pseudo functors F and G is an assignment $A \mapsto \alpha_A$ which is natural up to a coherence iso. This coherence iso satisfies coherence diagrams.

$$\begin{array}{ccc} FA & \xrightarrow{\alpha_A} & GA \\ {\scriptstyle Ff}\downarrow & \overset{\tau_f}{\swarrow\!\!\swarrow} & \downarrow{\scriptstyle Gf} \\ FB & \xrightarrow[\alpha_B]{} & GB \end{array}$$

Pseudo Limits

Let $F : \mathcal{J} \to \mathcal{C}$ be a pseudo functor. Then $L \in \mathcal{C}$ and $\pi \in PsCone(L, F)$ comprise a *pseudo limit* of F if

$$Mor_{\mathcal{C}}(L', L) \to PsCone(L', F)$$

$$f \mapsto \pi \circ f$$

is an isomorphism of categories for all $L' \in \mathcal{C}$.

Variants: *weighted pseudolimits*, *bilimits*, *weighted bilimits*

Example 2-equalizer vs. Pseudo-equalizer

Suppose $F : \{1 \rightrightarrows 2\} \to Cat$ has image

$$A \underset{H}{\overset{G}{\rightrightarrows}} B.$$

Then Obj 2-lim $F = \{a \in A : Ga = Ha\}$, while the object class of the pseudo limit $PsCone(\mathbf{1}, F)$ is

$$\{(a, b, g, h) : a \in A, b \in B, g : Ga \overset{\cong}{\longrightarrow} b, h : Ha \overset{\cong}{\longrightarrow} b\}.$$

Hence 2-lim F may be empty while ps-lim F is not.

Theories and Algebras

A *Lawvere theory* is a category T with objects $0, 1, 2, \ldots$ such that n is the product of n-copies of 1 and each n is equipped with a limiting cone.

Example The Endomorphism Theory

Let X be a set. The *endomorphism theory* $End(X)$ has objects $0, 1, 2, \ldots$ and morphisms $Hom_{End(X)}(m, n) = Maps(X^m, X^n)$.

A *theory* is a collection of sets $T(0), T(1), \ldots$ equipped with maps

$$\gamma : T(k) \times T(n_1) \times \cdots \times T(n_k) \to T(n_1 + \cdots + n_k)$$

that are associative, unital, and are equivariant with respect to subsitution.

Translation: $T(n) = T(n, 1)$ and also $T(m, n) = \prod_{i=1}^{n} T(m)$.

A small category X is a T-*algebra* or an *algebra over* T if it is equipped with a morphism of theories $(\Phi_n : T(n) \to End(X)(n))_n$.

Example Groups

The theory T of groups is generated by $e \in T(0), \nu \in T(1), \mu \in T(2)$ and these satisfy the usual axioms of a group. A group is an algebra over the theory of groups.

A small category X is a *pseudo T-algebra* if it is equipped with

- structure maps $\Phi_n : T(n) \to End(X)(n)$ for each $n \geq 0$

- coherence isos for each operation of theories (γ, 1, and substitution) which satisfy

- coherence diagrams for each relation of theories (associativity, unitality, equivariance).

For example, for $w, v, u \in T(1)$ the diagram

$$\begin{array}{ccc} \Phi_1(w) \circ \Phi_1(v) \circ \Phi_1(u) & \Longrightarrow & \Phi_1(w) \circ \Phi_1(v \circ u) \\ \Downarrow & & \Downarrow \\ \Phi_1(w \circ v) \circ \Phi_1(u) & \Longrightarrow & \Phi_1(w \circ v \circ u) \end{array}$$

must commute.

Example Finite sets with $\coprod$ form a pseudo commutative monoid.

$$A \coprod B :\!=\!= A \times \{1\} \cup B \times \{2\}$$

$$B \coprod A == B \times \{1\} \cup A \times \{2\}$$

Example Finite dimensional vector spaces with $\oplus$ and $\otimes$ form a pseudo commutative semiring.

Example Any symmetric monoidal category is a pseudo commutative monoid.

Theorem 1 *(F) Let T be a theory. Let $\mathcal{C}_T$ denote the 2-category of pseudo T-algebras. Then any pseudo functor $F : \mathcal{J} \to \mathcal{C}_T$ from a 1-category $\mathcal{J}$ to the 2-category of pseudo T-algebras admits all weighted pseudo limits and all weighted bicolimits.*

Proof: Let $L := PsCone(\mathbf{1}, U \circ F)$, where $U : \mathcal{C}_T \to Cat$ is the forgetful 2-functor. Then L has a pseudo T-algebra structure and it is the conical pseudo limit of F. Theorems from Street (1980) about cotensor products imply the result for conical pseudo limits. The conical bicolimits require a different construction. $\square$

This allows one to speak of stacks of pseudo T-algebras.

Theorem 2 *(F) Let T be a theory. There exists a 2-monad C on $\mathcal{C}at$ such that the 2-category of strict C-algebras, pseudo C-morphisms, and 2-cells is 2-equivalent to $\mathcal{C}_T$.*

Proof: The 2-monad C is constructed by taking the free theory on T and enriching it appropriately in groupoids. □

Theorem 3 *(Blackwell, Kelly, Power 1989) Let C be a 2-monad on a 2-category $\mathcal{K}$. Then the 2-category of strict C-algebras, pseudo C-morphisms, and 2-cells admits all strictly weighted pseudo limits of strict 2-functors.*

This theorem implies a special case of Theorem 1.

Bi-adjoints

Let $\mathcal{X}, \mathcal{A}$ be 2-categories.
A *bi-adjunction* $\langle F, G, \phi \rangle : \mathcal{X} \rightharpoonup \mathcal{A}$ consists of the following data

- Pseudo functors

$$\mathcal{X} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{A}$$

- For all $X \in Obj\mathcal{X}$ and all $A \in Obj\mathcal{A}$ an equivalence of categories $\phi_{X,A} : Mor_{\mathcal{A}}(FX, A) \to Mor_{\mathcal{X}}(X, GA)$ which is pseudo natural in X and A separately.

Theorem 4 *(Kelly 1989) (F) Let $\mathcal{X}, \mathcal{A}$ be 2-categories.*
Let $\mathcal{X} \xleftarrow{G} \mathcal{A}$ be a pseudo functor. Then there exists a left bi-adjoint for G if and only if for every object $X \in Obj\mathcal{X}$ there exists an object $R \in Obj\mathcal{A}$ and a bi-universal arrow
$\eta_X : X \to G(R)$ from X to G.

Theorem 5 *(F) Let $S \to T$ be a morphism of theories and let $G : \mathcal{C}_T \to \mathcal{C}_S$ be the associated forgetful 2-functor. Then there exists a left bi-adjoint $F : \mathcal{C}_S \to \mathcal{C}_T$ which is a strict 2-functor. Moreover, $\phi_{X,A}$ is strictly 2-natural in each variable.*

2-Theories, Algebras, and Pseudo Limits

A *2-theory* (Θ, T) *fibered over the theory* T is a natural number k, a theory T, and a contravariant functor $\Theta : T \to Cat$ from the category T to the 2-category Cat of small categories such that

- $Obj\Theta(m) = \coprod_{n \geq 0} Hom_{T^k}(m, n)$ for all $m \geq 0$, where T^k is the theory with the same objects as T, but with $Hom_{T^k}(m, n) = Hom_T(m, n)^k$

- If $w_1, \ldots, w_n \in Hom_{T^k}(m, 1)$, then the word in $Hom_{T^k}(m, n)$ with which the n-tuple is identified is the product in $\Theta(m)$ of $w_1, \ldots, w_n$

- For $w \in Hom_T(m, n)$ the functor $\Theta(w) : \Theta(n) \to \Theta(m)$ is $\Theta(w)(v) = v \circ w^{\times k}$ on objects $v \in Hom_{T^k}(n, j)$.

For objects $w_1, \ldots, w_n, w \in Mor_{T^k}(m, 1)$
$\subseteq Obj\Theta(m)$ we set

$$\Theta(w; w_1, \ldots, w_n) := Mor_{\Theta(m)}(\prod_{i=1}^{n} w_i, w).$$

Example Let I be a 1-category and $X : I^k \to Cat$ a strict 2-functor. Then there is a 2-functor $End(X) : End(I)^{op} \to Cat$ called the *endomorphism 2-theory on* X which is defined analogously to the endomorphism theory on a set.

A *pseudo* (Θ, T)-*algebra* consists of the following data:

- a small pseudo T-algebra I with structure maps denoted $\Phi : T(n) \to Functors(I^n, I)$

- a strict 2-functor $X : I^k \to Cat$

- set maps $\phi : \Theta(w; w_1, \ldots, w_n) \to End(X)(\Phi(w); \Phi(w_1), \ldots, \Phi(w_n))$, where $\Phi(w)$ means to apply Φ to each component of w to make I^k into the product pseudo algebra of k copies of I.

- a coherence iso for each operation of a 2-theory satisfying

- coherence diagrams for each relation of 2-theories.

Example The 2-theory of *commutative monoids with cancellation* (CMC's) is generated by 3 operations. The underlying theory T is commutative monoids and k=2. The operations are described below in terms of an algebra $X : I^2 \to Cat$ rather than in terms of the abstract 2-theory.

$$+ : X_{a,b} \times X_{c,d} \to X_{a+c,b+d}$$

$$0 \in X_{0,0}$$

$$\check{?} : X_{a+c,b+c} \to X_{a,b}$$

Axioms:
$+$ is commutative, associative, and unital
$\check{?}$ is transitive and distributive.

Example Rigged surfaces form a pseudo algebra over the 2-theory of CMC's.

$I :=$ Finite Sets and bijections equipped with a choice of $\coprod$
$X : I^2 \to Cat$
$X_{a,b} :=$ the category of rigged surfaces with inbound boundary components labelled by the finite set a and outbound boundary components labelled by the finite set b
$+ : X_{a,b} \times X_{c,d} \xrightarrow{\coprod} X_{a \coprod c, b \coprod d}$
$\check{?} : X_{a \coprod c, b \coprod c} \xrightarrow{\text{glue}} X_{a,b}$

Theorem 6 *(F) The 2-category of pseudo (Θ, T)-algebras admits weighted pseudo limits.*

This allows one to speak of stacks of pseudo (Θ, T)-algebras, in particular stacks of pseudo commutative monoids.

Definition of CFT

An *abstract chiral conformal field theory* on a stack of pseudo commutative monoids $\mathcal{D}$ is a Hilbert space $\mathcal{H}$ and a morphism $\phi : \mathcal{D} \to \underline{\mathcal{H}}$ of stacks of pseudo commutative monoids over the identity map of the underlying stack of pseudo commutative monoids.

This definition accounts for the coherence isomorphisms involved in disjoint union and gluing.

Conclusion

Certain bicategorical limits, colimits, and adjoints exist in the context of pseudo algebras over theories and 2-theories. Their existence allows one to speak of stacks in the definition of CFT formulated by Hu and Kriz.