MODULI SPACES, JACOBIANS, AND RIGGED SURFACES STUDENT GEOMETRY/TOPOLOGY SEMINAR TALK SEPTEMBER 24TH, 2004

THOMAS M. FIORE

1. Introduction

In this talk I will introduce the notions of moduli space and Jacobians and briefly describe a current research topic. The primary example is the moduli space of elliptic curves. The moduli space of higher dimensional principally polarized complex tori is introduced by way of analogy. The process of taking the Jacobian of a compact Riemann surface gives a functor from the category of compact Riemann surfaces to a discrete category made up of moduli spaces of tori. The research question Igor Kriz and I are investigating is: in what sense can this functor be extended on rigged surfaces to form a morphism of stacks of pseudo commutative monoids with cancellation? Since this is an expository talk, the treatment will be brief. Proofs will only be sketched, or omitted entirely.

2. Moduli Spaces

When we say the *moduli space* of something, we generally mean the collection of isomorphism classes of those things. One useful aspect of moduli space is that it usually has additional structure, *e.g.* it is a variety, a manifold, or a topological space. Here are some examples of decreasing triviality.

Example 1. The moduli space of groups of order 2 is the singleton, because any two groups of order 2 are isomorphic. The moduli space of groups of order three is also the singleton. The moduli space of groups of prime order is in bijective correspondence with the prime numbers, since any two groups of prime order p_1 and p_2 are isomorphic if and only if $p_1 = p_2$.

Example 2. The moduli space of topological spaces with 2 elements has cardinality 3. The classes are represented by the discrete topology, the topology with 2 open sets, and the topology with three open sets.

Example 3. The moduli space of finite dimensional complex vector spaces is $\mathbb{N} \cup \{0\}$, since two finite dimensional complex vector spaces are isomorphic if and only if they have the same dimension. Note that this moduli space has an algebraic structure, whereas the previous examples were not so good because they didn't have any additional structure.

Example 4. The moduli space of closed compact connected orientable 2 dimensional real manifolds is $\mathbb{N} \cup \{0\}$ because such surfaces are entirely classified by the genus, *i.e.* by their number of holes.

Date: November 2, 2004.

3. Elliptic Curves

Elliptic curves are curves that are important in algebraic geometry, number theory, and algebraic topology. See [8] for more on elliptic curves. They are important in algebraic topology because they relate to formal group laws and elliptic cohomology.

Definition 3.1. An *elliptic curve* is a subset of $\mathbb{C}P^2$ consisting of $[0:1:1]$ and all points $[x:y:1]$ satisfying the *Weierstrass equation*

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

for some $a_1, a_2, a_3, a_4, a_6 \in \mathbb{C}$.

Most authors intentionally leave out a_5 because it makes something else easier. Elliptic curves can also be described as 1 dimensional complex tori, and that is what we will in fact use. A 1-dimensional complex torus is just the complex plane modded out by a rank 2 lattice. Here a *rank 2 lattice* is the $\mathbb{Z}$-span of two $\mathbb{R}$-linearly independent complex numbers. In fact, when we say elliptic curve in this talk, we will usually think of the complex tori. The identification is in the following well known theorem.

Theorem 3.1. *The set of isomorphism classes of elliptic curves is in bijective correspondence with the set of isomorphism classes of genus 1 surfaces with complex structure. Moreover, these have the form $\mathbb{C}/\Lambda$ for lattices $\Lambda \subseteq \mathbb{C}$.*

Proof: We show how to get an elliptic curve from a complex torus. The Weierstrass p-function associated to the lattice Λ is

$$p(x;\Lambda) := \frac{1}{x^2} + \sum_{0\neq\omega\in\Lambda}\left(\frac{1}{(x-\omega)^2} - \frac{1}{\omega^2}\right).$$

It gives us an embedding

$$\begin{aligned} \mathbb{C}/\Lambda &\hookrightarrow \mathbb{C}P^2 \\ [x] &\mapsto [p(x;\Lambda) : \tfrac{1}{2}p'(x;\Lambda) : 1]. \end{aligned}$$

□

Example 5. The tori

$$\frac{\mathbb{C}}{\langle 1, i\rangle_{\mathbb{Z}}} \text{ and } \frac{\mathbb{C}}{\langle 1, e^{\pi i/3}\rangle_{\mathbb{Z}}}$$

are elliptic curves. Note that topologically they are the same, but as elliptic curves they are different. The right lattice is the honeycomb lattice.

4. The Moduli Space of Elliptic Curves

Sometimes people refer to moduli space as *parameter space*. We'll see why in the case of the moduli space of elliptic curves. In this part of the talk we briefly describe isomorphim classes of elliptic curves in terms of a parameter τ in the upper half plane $\mathbb{H} := \{z \in \mathbb{C} : \text{Im } z > 0\}$.

For the purposes of moduli space, it suffices to speak of lattices with basis vectors 1 and τ with $\tau \in \mathbb{H}$ because any lattice in $\mathbb{C}$ can be rotated and dilated to get one of this form. The rotated and dilated lattice gives rise to an elliptic curve isomorphic to the elliptic curve obtained from the original lattice. More precisely, *homothetic* lattices Λ and Λ' give rise to isomorphic elliptic curves $\mathbb{C}/\Lambda$ and $\mathbb{C}/\Lambda'$, where homothetic means that there exists a nonzero complex number z such that $\Lambda = z\Lambda'$.

However, it is still possible for elliptic curves

$$\frac{\mathbb{C}}{\langle 1, \tau \rangle_{\mathbb{Z}}} \text{ and } \frac{\mathbb{C}}{\langle 1, \tau' \rangle_{\mathbb{Z}}}$$

to be isomorphic although τ and τ' are distinct. For example, $\langle 1, \tau \rangle = \langle 1, \tau+1 \rangle$ give rise to equal elliptic curves although $\tau \neq \tau + 1$. Note also that $\langle 1, \tau \rangle$ and $\langle 1, -1/\tau \rangle$ give rise to isomorphic elliptic curves because of the homothety $z = -1/\tau$. It turns out that we get isomorphic elliptic curves only when τ an τ' are related by repeated applications of these two operations. The group of transformations $\mathbb{H} \to \mathbb{H}$ generated by

$$\tau \mapsto \tau + 1 \text{ and } \tau \mapsto -1/\tau$$

consists of all transformations of the form

$$\tau \mapsto \frac{a\tau + b}{c\tau + d} \text{ where } det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1 \text{ and } a, b, c, d \in \mathbb{Z},$$

which is isomorphic to the group $PSL(2, \mathbb{Z}) = SL(2, \mathbb{Z})/\{+1, -1\}$.

Theorem 4.1. *The moduli space $\mathcal{M}$ of elliptic curves is*

$$PSL(2, \mathbb{Z})\backslash\mathbb{H} = SL(2, \mathbb{Z})\backslash\mathbb{H}.$$

Proof: Consider the map $\phi : \mathbb{H} \to \mathcal{M}$ defined by

$$\tau \mapsto \text{isomorphism class of } \frac{\mathbb{C}}{\langle 1, \tau \rangle_{\mathbb{Z}}}.$$

By the above discussion, we have $\phi(\tau) = \phi(\tau')$ if and only if τ and τ' belong to the same $PSL(2, \mathbb{Z})$-orbit. This induces a bijection between $PSL(2, \mathbb{Z})\backslash\mathbb{H}$ and $\mathcal{M}$. □

One can draw a nice picture of this, as I did in the talk. The picture shows that this moduli space is itself a Riemann surface. This is the additional structure I mentioned in the general discussion of moduli spaces. I should also mention that elliptic curves are totally classified by their j-invariant, which gives us a bijection between $\mathcal{M}$ and $\mathbb{C}$. We see that this moduli space is topologically the complex plane. See [1].

5. The Moduli Space of Higher Dimensional Complex Tori

Although it is not obvious, the appropriate generalization of the upper half plane $\mathbb{H}$ is the *Siegel Upper Half Space* $\mathbb{H}_g$ of complex symmetric $g \times g$ matrices Ω with positive definite imaginary part. A g-dimensional complex torus is given by $\mathbb{C}^g/(\mathbb{Z}^g + \Omega\mathbb{Z}^g)$. Such tori appeared in the 19th century as Jacobian varieties of compact Riemann surfaces. The moduli space of g-dimensional principally polarized complex tori is

$$Sp(2g, \mathbb{Z})\backslash\mathbb{H}_g$$

where $Sp(2g, \mathbb{Z})$ acts on $\mathbb{H}_g$ by $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \Omega := (A\Omega + B)(C\Omega + D)^{-1}$. Note that $SL(2, \mathbb{Z}) = Sp(2, \mathbb{Z})$ so that this coincides with the moduli space of elliptic curves when $g = 1$. Mumford has described this moduli space, how it relates to Jacobians, and much, much more in [6].

6. Jacobians of Compact Riemann Surfaces

There is a map Jac from compact Riemann surfaces to varieties defined as follows. Let X be a genus g compact Riemann surface and $\Omega^1(X)$ its vector space of global holomorphic 1-forms. There is a biadditive map

$$H_1(X;\mathbb{Z}) \times \Omega^1(X) \to \mathbb{C}$$
$$([c],\omega) \mapsto \int_{[c]} \omega$$

given by integration. Any linear map $\Omega^1(X) \to \mathbb{C}$ that can be expressed as $\omega \mapsto \int_{[c]} \omega$ is called a *period*. The periods form a lattice $\Lambda \subset \Omega^1(X)^* \cong \mathbb{C}^g$. The *Jacobian* of X is

$$Jac(X) := \Omega^1(X)^*/\Lambda$$

and this can be expressed as $\mathbb{C}^g/(\mathbb{Z}^g + \Omega\mathbb{Z}^g)$ after choosing a basis appropriately. This is described thoroughly in [5] and [6].

Example 6. The Jacobian of $\mathbb{C}P^1$ is $\{0\}$ because $\Omega^1(\mathbb{C}P^1) = 0$.

Example 7. The Jacobian of an elliptic curve C is isomorphic to the elliptic curve C.

7. Jacobians of Rigged Surfaces

If we consider the category $\mathcal{C}_{cl}$ of closed compact Riemann surfaces with morphisms given by holomorphic diffeomorphisms and we view the moduli space as a discrete category, then Jac is a functor

$$\mathcal{C}_{cl} \to \bigcup_{g\geq 0} (Sp(2g,\mathbb{Z})\backslash \mathbb{H}_g).$$

Can this functor be extended to the larger category of rigged surfaces?

A *rigged surface* is a compact 2-dimensional manifold with complex structure equipped with analytic parametrizations of the boundary components. Examples of rigged surfaces are the Riemann sphere, any Riemann surface, and the unit disk. Morphisms of rigged surfaces are holomorphic diffeomorphisms which preserve the boundary parametrizations. Let $\mathcal{C}$ denote the category of rigged surfaces. It has a pseudo algebraic structure of gluing and disjoint union which makes it into a pseudo commutative monoid with cancellation as in [2],[3],[4]. Igor Kriz and I are currently working to expand the moduli category into one which admits gluing and disjoint union in such a way that the Jacobian functor extends to a morphism of pseudo commutative monoids. Lattice conformal field theories should factor through this generalized Jacobian functor. See [3] and [7] for the definition of conformal field theory.

References

[1] J. Baez: http://math.ucr.edu/home/baez/week197.html

[2] T. Fiore: *Pseudo Limits, Bi-Adjoints, and Pseudo Algebras: Categorical Foundations of Conformal Field Theory*, PhD Thesis, available at `http://arxiv.org/abs/math.CT/0408298`

[3] P. Hu and I. Kriz: Conformal Field Theory and Elliptic Cohomology, to appear in *Advances in Mathematics.*

[4] P. Hu and I. Kriz: Closed and Open Conformal Field Theories and Their Anomalies, *to appear.*

[5] R. Miranda: *Algebraic Curves and Riemann Surfaces*, Graduate Studies in Mathematics Volume 5, AMS, Providence, 1995.
[6] D. Mumford: *Tata Lectures on Theta I*, Progress in Mathematics 28, Birkhäuser, Boston, 1983.
[7] G. Segal: The Definition of Conformal Field Theory, *preprint* from the 1980's.
[8] J.H. Silverman: *The Arithmetic of Elliptic Curves*, Graduate Texts in Mathematics 106, Springer Verlag, New York, 1986.

Department of Mathematics, University of Michigan, Ann Arbor, MI, 48109
E-mail address: fioret@umich.edu