Mathematics and Music
2009 REU
Lecture 1 Problems: Introduction and Basic Scale Theory
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1. If a and b are pitches designated by their frequency, we write

$$a \sim b$$

if $a/b = 2^j$ for some $j \in \mathbb{Z}$, in other words if a and b are a whole number of *octaves* apart. Prove that this is an equivalence relation, in other words show
$a \sim a$ for all pitches a
$a \sim b$ implies $b \sim a$ for all pitches a and b
$a \sim b$ and $b \sim c$ implies $a \sim c$ for all pitches a, b, and c.

2. Recall the functions

$$T_n : \mathbb{Z}_{12} \to \mathbb{Z}_{12}$$
$$T_n(x) := x + n.$$

Apply T_5 repeatedly to 0 (remember to use arithmetic modulo 12). This is the *circle of fourths*. Why should you expect to get 12 pitch classes?

How can you use the attached circle of fifths to determine the key signature for the E major scale?

3. Apply T_1 repeatedly to the C major scale

$$\{C, D, E, F, G, A, B, C\}$$
$$\{0, 2, 4, 5, 7, 9, 11, 0\}.$$

Do you get 12 different sets? The step intervals for the major scale are 2-2-1-2-2-2-1. Can you use these step intervals (or some other structure of the major scale) to see that there are indeed 12 different major scales, without computing all 12? A major scale is named after its first note when it is written with the step intervals 2-2-1-2-2-2-1. For example, the major scale above is the C-major scale because $C = 0$.

4. Using an approach like in the previous problem, prove that there are 12 pentatonic scales. The F-major pentatonic scale

$$\{F, G, A, C, D, F\}$$
$$\{5, 7, 9, 0, 2, 5\}$$

has step intervals 2-2-3-2-3.

5. Prove that there are only two whole tone scales. The whole tone scale based on C is

$$\{C, D, E, F\sharp, G\sharp, A\sharp, C\}$$
$$\{0, 2, 4, 6, 8, 10, 0\}$$

and it has step intervals 2-2-2-2-2-2.

6. Prove that there are only three octatonic scales. The $D\flat$ *octatonic scale* is

$$\{D\flat, E\flat, E, F\sharp, G, A, B\flat, C, D\flat\}$$

$$\{1, 3, 4, 6, 7, 9, 10, 0, 1\}.$$

Its step intervals are 2-1-2-1-2-1-2-1.

7. A scale is said to be *generated* if it is obtained by an iteration of T_n for some n. Is the chromatic scale

$$\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 0\}$$

generated? Prove that the octatonic scale

$$\{D\flat, E\flat, E, F\sharp, G, A, B\flat, C, D\flat\}$$
$$\{1, 3, 4, 6, 7, 9, 10, 0, 1\}$$

is not generated.

8. A generated scale is said to be *well formed* if each generating interval spans the same number of scale steps. Prove that the major scale

$$\{C, D, E, F, G, A, B, C\}$$
$$\{0, 2, 4, 5, 7, 9, 11, 0\}$$

is well formed.

9. A scale is said to have the *Myhill Property* if each scale interval comes in two chromatic sizes. Show that the pentatonic scale

$$\{F, G, A, C, D, F\}$$
$$\{5, 7, 9, 0, 2, 5\}$$

satisfies the Myhill property.

10. A scale is said to be *maximally even* if each scale intervals comes in either one chromatic size or two chromatic sizes of consecutive integers. Is the C-major scale

$$\{C, D, E, F, G, A, B, C\}$$
$$\{0, 2, 4, 5, 7, 9, 11, 0\}$$

maximally even?