Mathematics and Music
2006 REU
Lecture 2 Problems
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1. If G is a group and X is a set, then a *(left) group action of* G *on* X is a function

$$G \times X \to X$$

such that

$$g_1(g_2x) = (g_1g_2)x$$
$$1_Gx = x$$

for all $g_1, g_2 \in G$ and $x \in X$. For $x, y \in X$, we write $x \sim y$ if there exists some $g \in G$ such that $gx = y$. Prove that this is an equivalence relation, in other words
$x \sim x$
$x \sim y$ implies $y \sim x$
$x \sim y$ and $y \sim z$ implies $x \sim z$
for all $x, y, z \in X$. An *equivalence class*

$$[x] = \{y \in X | y \sim x\}$$

is called an *orbit*.

2. Prove that the set of functions $\{T_n | n \in \mathbb{Z}_{12}\} \cup \{I_n | n \in \mathbb{Z}_{12}\}$ forms a group. We call this group the T/I group.

3. Prove that the T/I group acts on the set of subsets of $\mathbb{Z}_{12}$. The orbits under this action are called *set classes*. Note that an orbit is a set of sets, and an element of an element of an orbit is a pitch class. The next problem should clarify this point.

4. Write down the set class of $\{0, 4, 7\}$. Can you say what this orbit is in musical terms? (Hint: use the dictionary to translate the numbers back into pitch classes).

5. Write down the set class of the diminished seventh chord $\{0, 3, 6, 9\}$. Why does this set class have only three elements?

6. Calculate the interval multiset and the interval vector of $\{0, 4, 7\}$.

7. Calculate the interval multiset and interval vector of the whole tone scale $\{0, 2, 4, 6, 8, 10\}$. Calculate the interval multiset and interval vector of the whole tone scale $\{1, 3, 5, 7, 9, 11\}$. By the hexachord theorem, the two whole tone scales should have the same interval vector. Is there an elementary way to see this using symmetry, or invariance of the interval multiset under transposition?

8. Calculate the interval multiset and interval vector of $\{8, 7, 10\}$ and $\{6, 3, 2\}$. We talked about these sets in Schoenberg Opus 23, Number 1.