Mathematics and Music
2009 REU
Lecture 2 Problems: Scale Theory and Algebraic Combinatorics on Words
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1. (Words) Recall that a *word* in the alphabet $\{a, b\}$ is an element of the free monoid $\{a, b\}^*$ on the set $\{a, b\}$. Two words w and w' are said to be *conjugate* if there exist words u and v such that $w = uv$ and $w' = vu$. Give some examples of conjugate words. Prove that the conjugates of w are precisely the *rotations* of w. Prove that the words w and w' are conjugate if and only if they are conjugate in the *free group* on $\{a, b\}$, that is, if and only if there exists an element g in the free group on $\{a, b\}$ such that $w = gw'g^{-1}$.

2. (Christoffel words) Let p and q be relatively prime positive integers and let $n = p+q$. The *Christoffel word of slope* $\frac{p}{q}$ *and length* n is the word $w = w_1 w_2 \cdots w_n$ with

$$w_i = \begin{cases} a & \text{if} \quad p \cdot i \bmod n > p \cdot (i-1) \bmod n \\ b & \text{if} \quad p \cdot i \bmod n < p \cdot (i-1) \bmod n. \end{cases}$$

Calculate the Christoffel word of slope $\frac{1}{2}$ and the Christoffel word of slope $\frac{2}{5}$. Check that the latter is the step-interval pattern for the Lydian mode where a represents a whole step and b represents a half step. Recall that we also defined the Christoffel word of slope $\frac{p}{q}$ as the *lower discretization of the line* $y = \frac{p}{q}x$, that is, we approximate from below the line segment from $(0, 0)$ to $(q, 1)$ as closely as possible, by moving only horizontally and vertically, only touching the line at $(0, 0)$ and $(q, 1)$. A horizontal edge is labelled a and a vertical edge is labelled b. Draw the lower discretization for the lines of slope $\frac{1}{2}$ and $\frac{2}{5}$ and check that the resulting words are the ones you found above. Prove that the algebraic definition and the lower discretization definition are equivalent.

3. (Christoffel duality) If w is a Christoffel word of slope $\frac{p}{q}$, its *Christoffel dual* w^* is the Christoffel word w^* of slope $\frac{p^*}{q^*}$, where p^* and q^* are the multiplicative inverses of p and q in $\mathbb{Z}_{p+q}$. We like to write the Christoffel dual in the alphabet $\{x, y\}$. Find the Christoffel duals of the Christoffel words in problem 2. Check that the Christoffel dual of Lydian mode is the fifth-fourth folding of the Lydian mode, where x represents "up by fifth" and y represents "down by a fourth". In this particular instance, we begin the folding on the first note of the Lydian, so on the note F if we're thinking in the key of C.

4. (Tetractys and its bad conjugate) Write out the conjugates of the Christoffel word aab. In the lecture we had a theorem of Clampitt-Domínguez-Noll: a word r is a conjugate of a Christoffel word if and only if $r = F(ab)$ for some $F \in St$. The monoid St is the monoid of *Sturmian morphisms*. These are all monoid homomorphisms $\{a, b\}^* \to \{a, b\}^*$ that can be written as composites of $G, \tilde{G}, D, \tilde{D}, E$. Calculate

$$G(ab) \quad \tilde{G}(ab) \quad D(ab) \quad \tilde{D}(ab) \quad GE(ab) \quad \tilde{G}E(ab) \quad DE(ab) \quad \tilde{D}E(ab).$$

Find the conjugates of aab in this list, as guaranteed by the theorem. Why are there words in this list that are not conjugates of aab? In general, for a Christoffel word w of length n, there are n conjugates of w, and $n-1$ of these can be written as $F(ab)$ for some $F \in St_0$. The monoid St_0 is the monoid of *special Sturmian morphisms*. These are all monoid homomorphisms $\{a,b\}^* \to \{a,b\}^*$ that can be written as composites of $G, \tilde{G}, D, \tilde{D}$. Note E is not included. The unique conjugate of w which cannot be written as $F(ab)$ for some $F \in St_0$ is called the *bad conjugate of* w. Which is the *bad conjugate* of the Christoffel word aab? (You've already calculated everything you need.) It is a really interesting fact that the bad conjugate of the Lydian mode is the Locrian mode, namely the unique mode with the least relevance in music, considered by many to be a theoretical artifact (see Noll 2008, pages 88 and 94).

5. (Tetractys and divider incidence) Recall the Christoffel word aab and its conjugates from problem 4. Recall the theorem from Clampitt-Domínguez-Noll that the plain adjoints of certain Christoffel conjugates of the form $F(ab)$ can be calculated as $F^{rev}(xy)$. Use this theorem, and the fact that there is a bijection of conjugates, to calculate all plain adjoints of all conjugates of aab. Keep track of the dividers. Find by direct comparison the unique conjugate r of aab whose divider note is the same as the divider note of its plain adjoint $r^\square$ (for this you may want to actually look at notes realizing the tetractys, for example $\{G, C, F\}$ where a means go up a perfect fourth and b means go up a whole tone). Next use the Divider Incidence Theorem of Clampitt-Domínguez-Noll to confirm your finding. Recall that the Divider Incidence Theorem distinguishes the Ionian mode in the case of the diatonic scale. Which of the tetractys conjugates might you call "Ionian"?