Mathematics and Music
2006 REU
Lecture 3 Problems
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1. A group action is *transitive* if for any $x, y \in X$ there exists a $g \in G$ such that $gx = y$, in other words there is only one orbit and that orbit is all of X. A group action is *simple* or *free* if

$$gx = g'x \text{ for some } x \in X \Rightarrow g = g'.$$

A group action is *simply transitive* if it is both transitive and simple. Show that any group acts simply transitively on itself via left multiplication.

2. If a group G acts on a set X, then the *stabilizer of* $x \in X$ is the group

$$G_x := \{g \in G | gx = x\}.$$

The Orbit-Stabilizer Theorem says the following.

Theorem 0.1. *If a finite group G acts on a finite set X, then*

$$|G|/|G_x| = |\text{orbit of } x|$$

where $|\cdot|$ means "number of elements."

The finiteness assumption can be left off by changing the statement into a bijection between cosets and equivalence classes, but we only need the finite case for now. Use the Orbit-Stabilizer Theorem to show that if a finite group G acts simply transitively on a finite set X, then $|X| = |G|$. How does this apply to our analysis of Hindemith, *Ludus Tonalis*, Fugue in G? Does the converse hold? More specifically, does $|X| = |G|$ imply that G acts simply transitively?

3. Recall that the musical space S in Hindemith, *Ludus Tonalis*, Fugue in G consists of the transposed and inverted forms of $\langle G, C, D\rangle = \langle 7, 0, 2\rangle$. The transformation $J : S \to S$ is defined by:
$J(x) :=$ that form of opposite type as x that has the same first two pitch classes but in the opposite order.
Use the table of S below to calculate

$$J\langle 3, 8, 10\rangle$$

$$J\langle 5, 10, 0\rangle$$

$$J\langle 9, 2, 4\rangle$$

$$J\langle 8, 3, 1\rangle$$

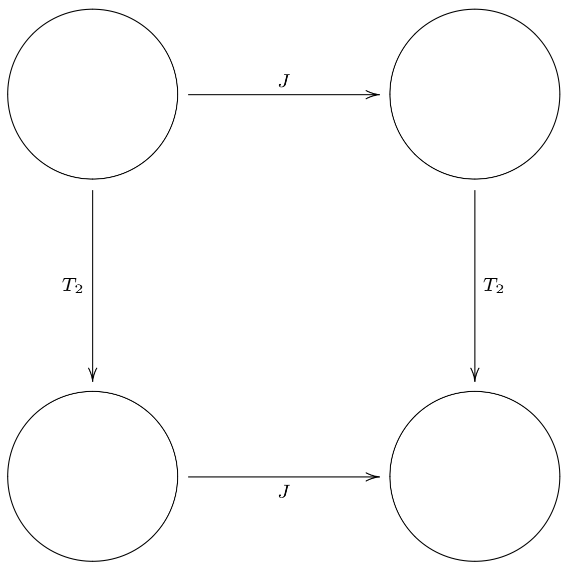
$$J\langle 10, 5, 3\rangle$$

$$J\langle 2, 9, 7\rangle.$$

Compare the answers for the first three with the answers for the second three. Do you see a pattern? Use this pattern to figure out $J \circ J(x)$.

Transposed Forms	Inverted Forms
$\langle 7, 0, 2\rangle$	$\langle 5, 0, 10\rangle$
$\langle 8, 1, 3\rangle$	$\langle 6, 1, 11\rangle$
$\langle 9, 2, 4\rangle$	$\langle 7, 2, 0\rangle$
$\langle 10, 3, 5\rangle$	$\langle 8, 3, 1\rangle$
$\langle 11, 4, 6\rangle$	$\langle 9, 4, 2\rangle$
$\langle 0, 5, 7\rangle$	$\langle 10, 5, 3\rangle$
$\langle 1, 6, 8\rangle$	$\langle 11, 6, 4\rangle$
$\langle 2, 7, 9\rangle$	$\langle 0, 7, 5\rangle$
$\langle 3, 8, 10\rangle$	$\langle 1, 8, 6\rangle$
$\langle 4, 9, 11\rangle$	$\langle 2, 9, 7\rangle$
$\langle 5, 10, 0\rangle$	$\langle 3, 10, 8\rangle$
$\langle 6, 11, 1\rangle$	$\langle 4, 11, 9\rangle$

4. Insert $\langle 0, 5, 7\rangle$ into the upper left circle of the *network*



and verify that the diagram commutes. This diagram can be found in measures 62 and 64 of Hindemith, *Ludus Tonalis*, Fugue in G.