Mathematics and Music
2009 REU
Lecture 4 Problems: Vuza Canons
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1. (Tilings, factorizations, rhythmic canons) Let A be a subset of $\mathbb{Z}_n$. We say that A *tiles* $\mathbb{Z}_n$ if there exists a subset B of $\mathbb{Z}_n$ such that

$$\mathbb{Z}_n = \bigcup_{b \in B} A + b$$

and $(A+b) \cap (A+b') = \emptyset$ for all $b, b' \in B$ with $b \neq b'$. In others words, $\mathbb{Z}_n$ is a disjoint union of translates of A. In this case, we write $\mathbb{Z}_n = A \oplus B$ and call this a *factorization of* $\mathbb{Z}_n$. A *rhythmic canon* is a factorization $\mathbb{Z}_n = A \oplus B$ and A is called the *inner rhythm* and B is called the *outer rhythm*. Let $n = 8$. Does $\{0, 2\}$ tile $\mathbb{Z}_8$? How about $\{0, 3\}$, does it tile $\mathbb{Z}_8$? Let $n = 7$. List all tilings of $\mathbb{Z}_7$.

2. (Tilings, factorizations, rhythmic canons, and unique decompositions) Prove that the definition of factorization of $\mathbb{Z}_n$ is equivalent to saying that every element z of $\mathbb{Z}_n$ has a unique decomposition $z = a + b$ with $a \in A$ and $b \in B$.

3. (Periodicity) A subset A of $\mathbb{Z}_n$ is called *periodic* if there exists an element $z \in \mathbb{Z}_n$ such that $A + z = A$. A rhythmic cannon $\mathbb{Z}_n = A \oplus B$ is called *periodic* if either A or B is periodic. Which of the rhythmic canons in Problem 1 are periodic? Prove your answer, and recall Sands' identification of cyclic Hajós groups.

4. (Bad groups) A group G is called *bad* if the Hajós conjecture does not hold for G. Use the corollary to Sands' identification of cyclic Hajós groups to prove that $\mathbb{Z}_{72}$ is the smallest bad group.

5. (Audio samples on Fripertinger's site) Go to my music website and follow the link to Fripertinger's website on the Enumeration of Vuza Canons of Length 72 and 108 and listen to the audio files. Find some Vuza canons that you think sound really neat!