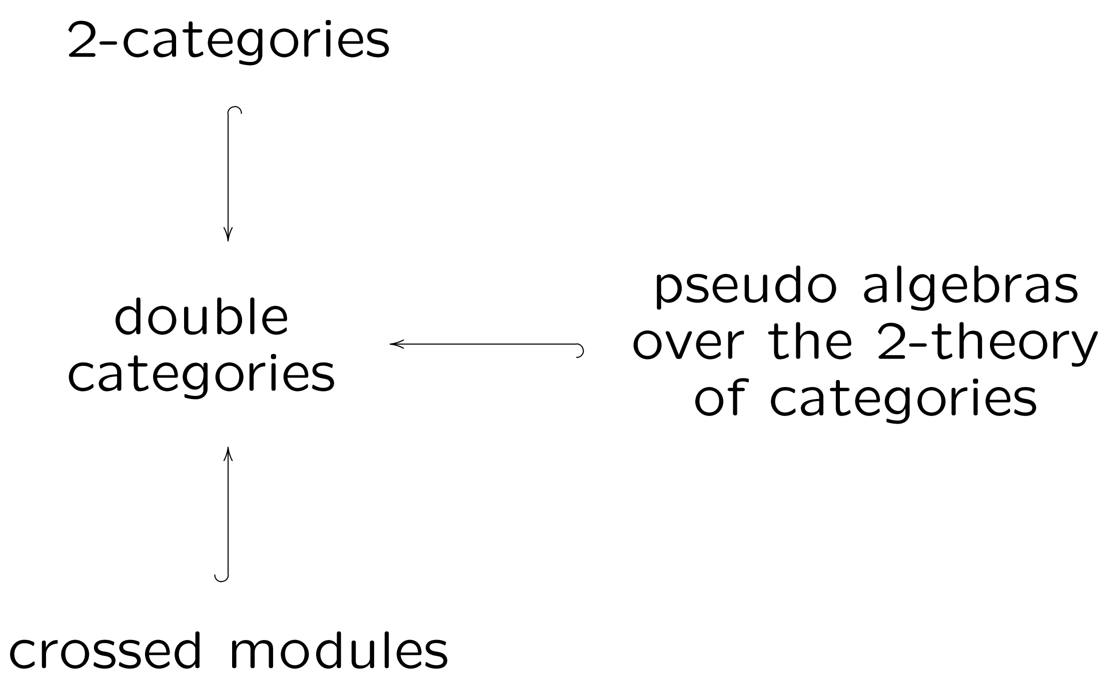
# Double Categories and Pseudo Algebras

Tom Fiore

www.math.uchicago.edu/∼fiore/

# Overview


2-categories
double categories
pseudo algebras over the 2-theory of categories
crossed modules

# Chronology

1942-1945 Eilenberg-Mac Lane: category theory

1946, 1950 Whitehead-Mac Lane: crossed modules, homotopy 2-types

1963 Ehresmann: double categories
Lawvere: Theories

1970's R. Brown: 2-groups, crossed modules, Van Kampen Theorems

1988 Segal Bourbaki talk: a CFT "is" a cocycle for elliptic cohomology

1991 Mac Lane: coherence in CFT

2002-2005 Fiore, Hu, Kriz: pseudo algebras over theories and 2-theories as a rigorous foundation of CFT

# 2-Categories

**Definition 1** *A 2-category* $\mathbf{C}$ *is a category enriched in categories, i.e.*

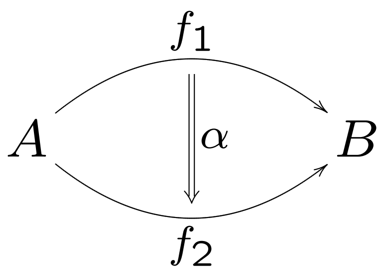
- *a set of objects* $Obj\,\mathbf{C}$
- *for each object* $A$ *and* $B$ *a category* $Mor_{\mathbf{C}}(A,B)$
- *composition functors*

  $$Mor_{\mathbf{C}}(B,C) \times Mor_{\mathbf{C}}(A,B) \xrightarrow{\circ} Mor_{\mathbf{C}}(A,C)$$

- *identities* $1_A \in Mor_{\mathbf{C}}(A,A)$

*which satisfy the usual axioms for a category.*

# Examples




**Example 1** *Any category is a 2-category with discrete morphism categories.*

**Example 2** *Topological spaces, continuous maps, homotopy classes of homotopies.*

**Example 3** *Categories, functors, and natural transformations form the 2-category $Cat$.*

**Example 4** *Rings, bimodules, bimodule maps form a bicategory.*

# Double Categories

**Definition 2** *(Ehresmann 1963) A double category* $\mathbb{D}$ *is an internal category in* $\mathcal{C}at$*.*

**Definition 3** *A double category* $\mathbb{D}$ *consists of a set of objects,*
*a set of horizontal morphisms,*
*a set of vertical morphisms, and*
*a class of squares with source and target as follows*

$$A \xrightarrow{f} B \qquad \begin{array}{c} A \\ {\scriptstyle j}\downarrow \\ C \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle j}\downarrow & \alpha & \downarrow{\scriptstyle k} \\ C & \xrightarrow[g]{} & D \end{array}$$

*and compositions and units that satisfy axioms.*

# Compositions and Units for Morphisms in a Double Category

Horizontal:

$$A \xrightarrow{f_1} B \xrightarrow{f_2} C = \begin{bmatrix} f_1 & f_2 \end{bmatrix} = f_2 \circ f_1$$

$$A \xrightarrow{1^h_A} A \xrightarrow{f_1} B = f_1 = A \xrightarrow{f_1} B \xrightarrow{1^h_B} B$$

Vertical:

$$\begin{array}{c} A \\ \downarrow{\scriptstyle j_1} \\ B \\ \downarrow{\scriptstyle j_2} \\ C \end{array} = \begin{bmatrix} j_1 \\ j_2 \end{bmatrix} = j_2 \circ j_1$$

$$\begin{array}{c} A \\ \downarrow{\scriptstyle 1^v_A} \\ A \\ \downarrow{\scriptstyle j_1} \\ B \end{array} = \begin{array}{c} A \\ \downarrow{\scriptstyle j_1} \\ B \end{array} = \begin{array}{c} A \\ \downarrow{\scriptstyle j_1} \\ B \\ \downarrow{\scriptstyle 1^v_B} \\ B \end{array}$$

# Compositions for Squares in a Double Category

Horizontal:

$$\begin{array}{ccccc} A & \xrightarrow{f_1} & B & \xrightarrow{f_2} & C \\ j\big\downarrow & \alpha & \big\downarrow k & \beta & \big\downarrow \ell \\ D & \xrightarrow[g_1]{} & E & \xrightarrow[g_2]{} & F \end{array} \quad = \quad \begin{array}{ccc} A & \xrightarrow{[f_1 f_2]} & C \\ j\big\downarrow & [\alpha\beta] & \big\downarrow \ell \\ D & \xrightarrow[[g_1 g_2]]{} & F \end{array}$$

Vertical:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ j_1\big\downarrow & \alpha & \big\downarrow k_1 \\ C & \xrightarrow{g} & D \\ j_2\big\downarrow & \beta & \big\downarrow k_2 \\ E & \xrightarrow[h]{} & F \end{array} \quad = \quad \begin{array}{ccc} A & \xrightarrow{f} & B \\ \left[\begin{smallmatrix} j_1 \\ j_2 \end{smallmatrix}\right]\big\downarrow & \left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right] & \big\downarrow \left[\begin{smallmatrix} k_1 \\ k_2 \end{smallmatrix}\right] \\ E & \xrightarrow[h]{} & F \end{array}$$

# Units for Squares in a Double Category

Horizontal:

$$
\begin{array}{ccccc}
A & \xrightarrow{1_A^h} & A & \xrightarrow{f} & B \\
{\scriptstyle j}\downarrow & i_j^h & {\scriptstyle j}\downarrow & \alpha & \downarrow{\scriptstyle k} \\
C & \xrightarrow[1_C^h]{} & C & \xrightarrow[g]{} & D
\end{array}
=
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
{\scriptstyle j}\downarrow & \alpha & \downarrow{\scriptstyle k} \\
C & \xrightarrow[g]{} & D
\end{array}
$$

$$
=
\begin{array}{ccccc}
A & \xrightarrow{f} & B & \xrightarrow{1_B^h} & B \\
{\scriptstyle j}\downarrow & \alpha & {\scriptstyle k}\downarrow & i_k^h & \downarrow{\scriptstyle k} \\
C & \xrightarrow[g]{} & D & \xrightarrow[1_D^h]{} & D
\end{array}
$$

Vertical:

$$
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
{\scriptstyle 1_A^v}\downarrow & i_f^v & \downarrow{\scriptstyle 1_B^v} \\
A & \xrightarrow{f} & B \\
{\scriptstyle j}\downarrow & \alpha & \downarrow{\scriptstyle k} \\
C & \xrightarrow[g]{} & D
\end{array}
=
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
{\scriptstyle j}\downarrow & \alpha & \downarrow{\scriptstyle k} \\
C & \xrightarrow[g]{} & D
\end{array}
=
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
{\scriptstyle j}\downarrow & \alpha & \downarrow{\scriptstyle k} \\
C & \xrightarrow{g} & D \\
{\scriptstyle 1_C^v}\downarrow & i_g^v & \downarrow{\scriptstyle 1_D^v} \\
C & \xrightarrow[g]{} & D
\end{array}
$$

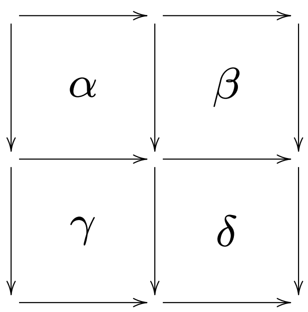
# Axioms for a Double Category

All compositions are *associative* and *unital* (as above) and

$$\begin{bmatrix} i^h_{j_1} \\ i^h_{j_2} \end{bmatrix} = i^h_{\left[\substack{j_1 \\ j_2}\right]}$$

$$\begin{bmatrix} i^v_{f_1} & i^v_{f_2} \end{bmatrix} = i^v_{[f_1 f_2]}.$$

*Interchange Law:*

If 

, then $\begin{bmatrix} \begin{bmatrix} \alpha & \beta \end{bmatrix} \\ \begin{bmatrix} \gamma & \delta \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \alpha \\ \gamma \end{bmatrix} & \begin{bmatrix} \beta \\ \delta \end{bmatrix} \end{bmatrix}$ and

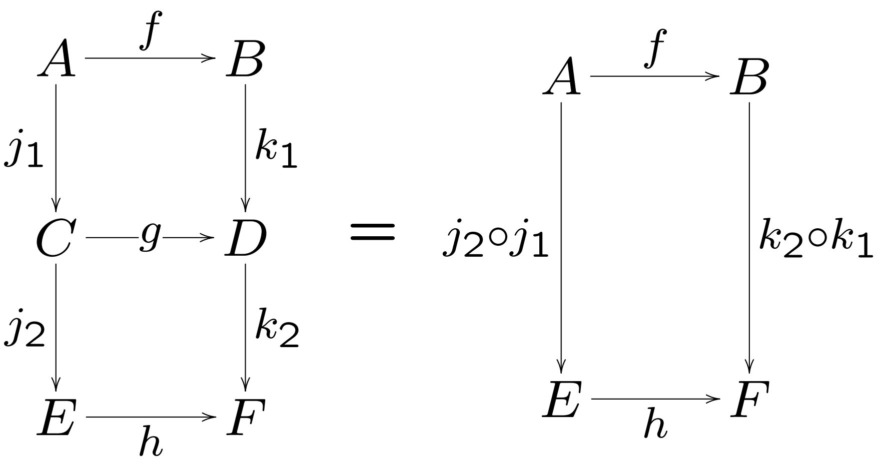
we write $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$.

# Examples of Double Categories

Let $I$ be a 1-category.

$\Box I$ := double category of commutative squares in $I$

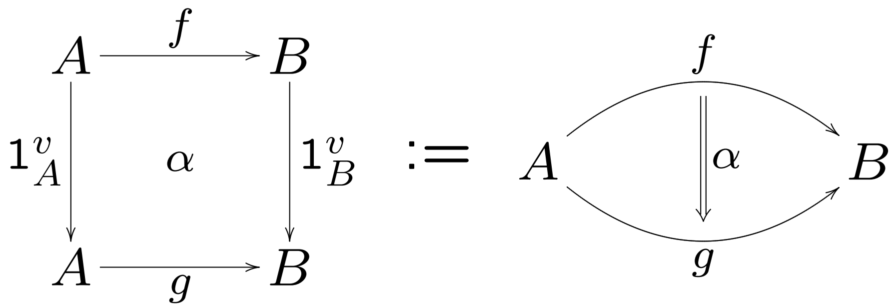
$Obj\,\Box I := Obj\, I$
$Hor\,\Box I := Mor\, I$
$Ver\,\Box I := Mor\, I$
$Sq\,\Box I :=$ commutative squares in $I$



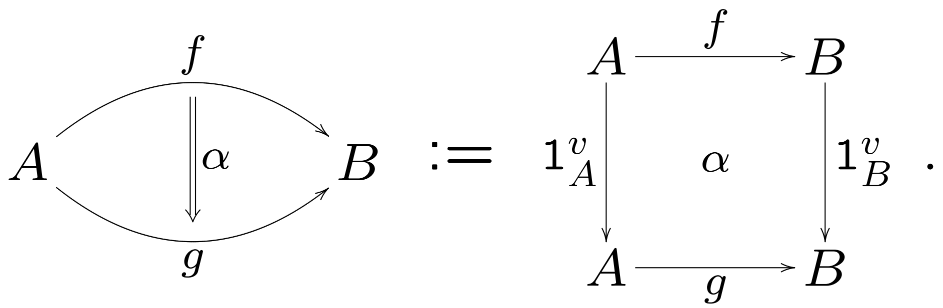

$\boxdot I$ := double category of not necessarily commutative squares in $I$

# Examples of Double Categories

Every 2-category $\mathbf{C}$ is a double category with trivial vertical morphisms.




**Definition 4** *The horizontal 2-category* $\mathbf{H}\mathbb{D}$ *of a double category* $\mathbb{D}$ *has objects* $Obj\,\mathbb{D}$*, morphisms* $Hor\,\mathbb{D}$*, and 2-cells*



# Examples of Double Categories

Let $\mathbf{C}$ be a 2-category.

$\mathbb{Q}\mathbf{C}$ := Ehresmann's double category of *quintets* in $\mathbf{C}$ (1963)

$Obj\,\mathbb{Q}\mathbf{C} := Obj\,\mathbf{C}$
$Hor\,\mathbb{Q}\mathbf{C} := Mor\,\mathbf{C}$
$Ver\,\mathbb{Q}\mathbf{C} := Mor\,\mathbf{C}$

$$Sq\,\mathbb{Q}\mathbf{C} := \left\{ \begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle j}\downarrow & \alpha & \downarrow{\scriptstyle k} \\ C & \xrightarrow[g]{} & D \end{array} \;\middle|\; A \;\underset{g\circ j}{\overset{k\circ f}{\Downarrow\alpha}}\; D \right\}$$

**Theorem 1** *(Grandis-Paré 2004) The functor* $\mathbb{Q} : 2\text{-}Cat \longrightarrow Dbl$ *admits a right adjoint.*

# Examples of Double Categories

$\mathbb{R}$ng:= pseudo double category or rings, bi-modules, and equivariant maps

$Obj\,\mathbb{R}$ng := rings with identity
$Hor\,\mathbb{R}$ng := bimodules
$Ver\,\mathbb{R}$ng := homomorphisms of rings
$Sq\,\mathbb{R}$ng :=

$$\left\{ \begin{array}{ccc} R & \xrightarrow{M} & S \\ {\scriptstyle j}\downarrow & \alpha & \downarrow{\scriptstyle k} \\ T & \xrightarrow[N]{} & U \end{array} \;\middle|\; \begin{array}{l} \alpha : M \longrightarrow N \text{ group homomorphism} \\ \alpha(smr) = k(s)\alpha(m)j(r) \end{array} \right\}$$

# Examples of Double Categories

Let $C$ be a topological category, *i.e.* $Obj\, C$ and $Mor\, C$ are topological spaces.

$\mathbb{P}'C$:= double category of Moore paths on $C$.

$Obj\, \mathbb{P}'C := Obj\, C$
$Hor\, \mathbb{P}'C := Mor\, C$
$Ver\, \mathbb{P}'C := P'(Obj\, C)$ = Moore paths in $Obj\, C$
$Sq\, \mathbb{P}'C := P'(Mor\, C)$ = Moore paths in $Mor\, C$

$P'X := \{(w, s) : s \geq 0, w : [0, s] \longrightarrow X\}$

# Examples of Double Categories

A *worldsheet* is a real, compact, not necessarily connected, two dimensional, smooth manifold with complex structure and real analytically parametrized boundary components.

$\mathbb{W}$:= pseudo double category of worldsheets

$Obj\,\mathbb{W}$ := finite sets
$Hor\,\mathbb{W}(A,B)$ := worldsheets with inbound components labelled by $A$ and outbound components by $B$
$Ver\,\mathbb{W}$ := bijections of finite sets
$Sq\,\mathbb{W}$ :=

$$\left\{ \begin{array}{ccc} A & \xrightarrow{x} & B \\ {\scriptstyle j}\downarrow & \alpha & \downarrow{\scriptstyle k} \\ C & \xrightarrow[y]{} & D \end{array} \;\middle|\; \begin{array}{l} \alpha : x \longrightarrow y \text{ holomorphic diffeo.} \\ \alpha \text{ compatible with } j \text{ and } k \\ \alpha \text{ preserves boundary params.} \end{array} \right\}$$

# Folding Structures

We introduce folding structures to compare algebras over the 2-theory of categories with double categories.

**Definition 5** *A holonomy on a double category* $\mathbb{D}$ *is a 2-functor*

$$(\mathbf{V}\mathbb{D})_0 \longrightarrow \mathbf{H}\mathbb{D}$$

$$A \longmapsto \overline{A} = A$$

$$\begin{array}{c} A \\ \downarrow j \\ B \end{array} \longmapsto A \xrightarrow{\overline{j}} B$$

**Example 5** *For a topological category* $C$*, a holonomy*

$$(\mathbf{V}\mathbb{P}'C)_0 \longrightarrow \mathbf{H}\mathbb{P}'C$$

*assigns to a path of objects a morphism from the initial point to the terminal point, like in differential geometry.*

# Folding Structures

**Definition 6** *A folding structure on a double category* $\mathbb{D}$ *consists of a holonomy* $j \mapsto \overline{j}$ *and bijections*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle j}\downarrow & \alpha & \downarrow{\scriptstyle k} \\ C & \xrightarrow[g]{} & D \end{array} \quad \overset{\Lambda}{\longleftrightarrow} \quad \begin{array}{ccc} A & \xrightarrow{[f\overline{k}]} & D \\ {\scriptstyle 1_A^v}\downarrow & \Lambda(\alpha) & \downarrow{\scriptstyle 1_D^v} \\ A & \xrightarrow[{[\overline{j}g]}]{} & D \end{array}$$

*compatible with compositions and units.*

A folding structure *horizontalizes* a double category.

# Examples of Folding Structures

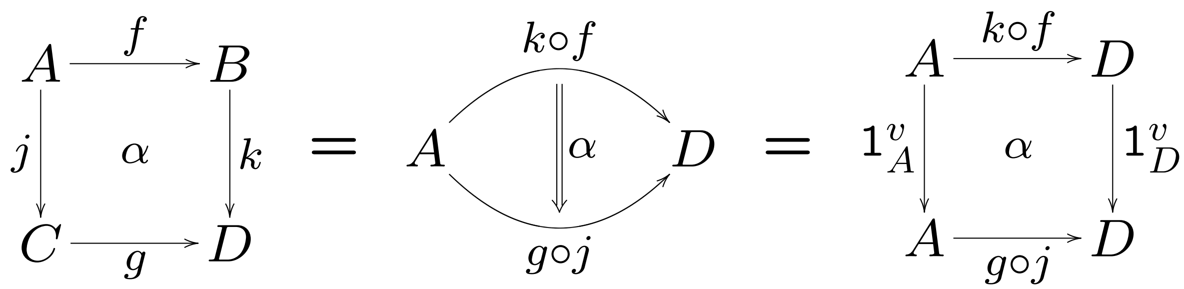
Let $I$ be a 1-category.
$\square I$=double category of commutative squares in $I$
$\boxdot I$=double category of not necessarily commutative squares in $I$

Then $\square I$ and $\boxdot I$ each admit a unique folding structure.

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle k} \\ C & \xrightarrow[g]{} & D \end{array} \quad \overset{\Lambda}{\longleftrightarrow} \quad \begin{array}{ccc} A & \xrightarrow{k\circ f} & D \\ {\scriptstyle 1^v_A}\downarrow & & \downarrow{\scriptstyle 1^v_D} \\ A & \xrightarrow[g\circ j]{} & D \end{array}$$

# Examples of Folding Structures

Let $\mathbf{C}$ be a 2-category. Then $\mathbb{Q}\mathbf{C}$ admits a folding structure by definition.



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Holonomy:

$$\overline{j} := T_j = \text{ the } (T,R)\text{-module } T$$

Folding:

$$\Lambda(\alpha) : U_k \bigotimes_S M \Longrightarrow N \bigotimes_T T_j$$

$$u \otimes m \longmapsto (u \cdot \alpha(m)) \otimes 1_T$$

# Examples of Folding Structures

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Holonomy:

$$\text{bijection} \quad \longmapsto \quad \begin{array}{c}\text{labelled union of} \\ \text{infinitely thin annuli}\end{array}$$

Folding:

relabel $x$ and $y$

# Comparison Theorems

**Observation 2** *(Brown-Mosa 1999, F. 2006) The notions of folding structure and connection pair are equivalent.*

**Theorem 3** *(F. 2006) The 2-category of strict 2-algebras over the 2-theory of categories is 2-equivalent to the 2-category of double categories with folding structures and invertible vertical morphisms.*

The pseudo version of the theorem also holds.

# Strict 2-Algebras

**Definition 7** *A strict 2-algebra over the 2-theory of of categories is a groupoid $I$ and a strict 2-functor $X : I^2 \longrightarrow Cat$ with strictly 2-natural functors*

$$X_{B,C} \times X_{A,B} \xrightarrow{\circ} X_{A,C}$$

$$\{*\} \xrightarrow{1_B} X_{B,B}$$

*for all $A, B, C \in I$ which satisfy axioms like those of a category.*

# Towards Strict 2-Algebras and Crossed Modules

Consider one object cases.

| | | |
|---|---|---|
| groupoids | $\subseteq$ 2-groupoids | $\subseteq$ double groupoids |
| groups | $\subseteq$ 2-groups | $\subseteq$ double groups |

**Theorem 4** *(Verdier, Brown-Spencer 1976,...) 2-groups are equivalent to crossed modules.*

**Theorem 5** *(Brown-Spencer 1976) Edge symmetric double groups with folding structure with trivial holonomy are equivalent to crossed modules.*

Question: What is a one object strict 2-algebra over the 2-theory of categories with everything iso?

# Crossed Modules

**Definition 8** *A crossed module is a group homomorphism* $\partial : H \longrightarrow G$ *with a left action*

$$G \times H \longrightarrow H$$

$$(g, \alpha) \mapsto {}^{g}\alpha$$

*by automorphisms such that:*

1. $\partial({}^{g}\alpha) = g\partial(\alpha)g^{-1}$ *for all* $\alpha \in H$ *and* $g \in G$

2. ${}^{\partial(\alpha)}\alpha_1 = \alpha\alpha_1\alpha^{-1}$ *for all* $\alpha, \alpha_1 \in H$.

**Example 6** $H \lhd G$ *is a crossed module*

# Crossed Modules

**Example 7** *Let* $(X, A, *)$ *be a based pair of spaces. Then*

$$\partial : \pi_2(X, A, *) \longrightarrow \pi_1(A, *)$$

*is a crossed module.*

**Theorem 6** *(Mac Lane-Whitehead 1950) Crossed modules model homotopy 2-types via this example.*

# Crossed Modules with Group Action

**Definition 9** *Let* $\partial : H \longrightarrow G$ *be a crossed module and* $I$ *a group. An action of* $I$ *on the crossed module* $\partial : H \longrightarrow G$ *consists of*

- *a left action of* $I$ *on* $H$ *by automorphisms written* $(j, \alpha) \mapsto {}^{j}\alpha$
- *a left action of* $I$ *on* $G$ *as a set written* $(j, g) \mapsto jg$
- *a right action of* $I$ *on* $G$ *as a set written* $(g, j) \mapsto gj$

# Crossed Modules with Group Action

These actions satisfy the following axioms for all $j, k \in I, \alpha \in H, g, g_1, g_2 \in G$.

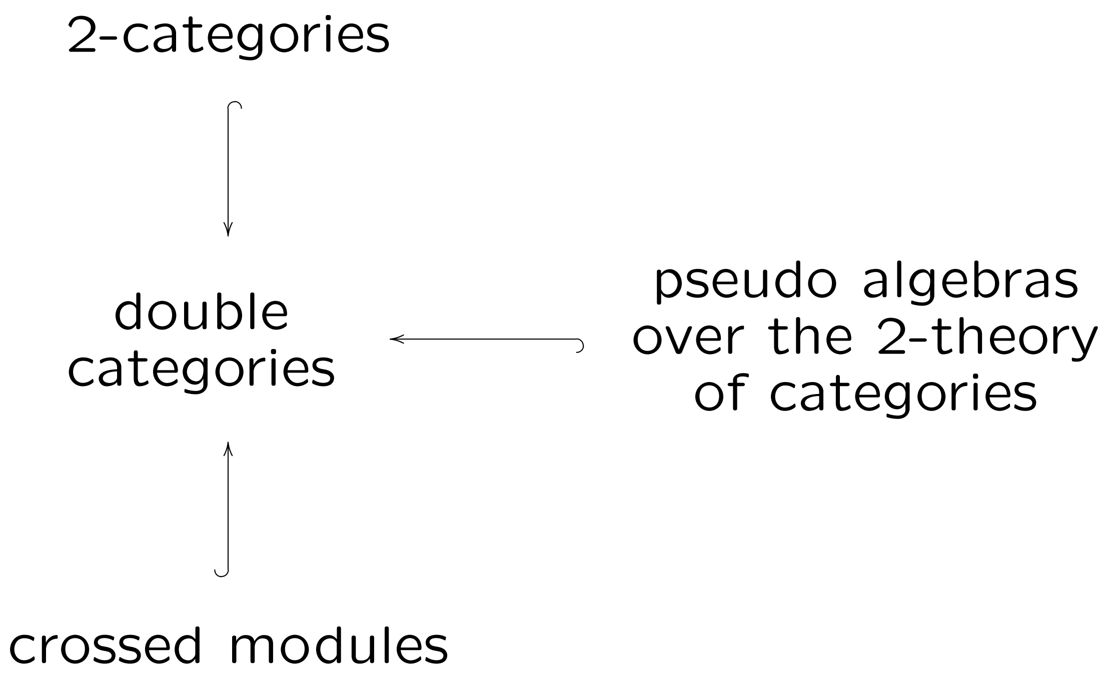
1. $(jg)k = j(gk)$
2. $(jg_1)g_2 = j(g_1g_2)$
3. $(g_1g_2)k = g_1(g_2k)$
4. $(g_1j)g_2 = g_1(jg_2)$
5. $j1_G = 1_Gj$
6. ${}^{(gk)}\alpha = {}^{g}({}^{k}\alpha)$
7. ${}^{(jg)}\alpha = {}^{j}({}^{g}\alpha)$
8. $\partial({}^{j}\alpha) = j\partial(\alpha)j^{-1}$

# Strict 2-Algebras and Crossed Modules

**Theorem 7** *(F. 2006) One object strict 2-algebras over the 2-theory of categories with everything iso are equivalent to crossed modules with a group action.*

In the case of a trivial group and trivial $I$, this says 2-groups are equivalent to crossed modules.

# Conclusion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