Groups Actions in neo-Riemannian Music Theory

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Introduction

- Mathematics is a very powerful descriptive tool in the physical sciences.
- Similarly, musicians use mathematics to communicate ideas about music.
- In this talk, we will discuss some mathematics commonly used by musicians.

Our Focus

Mathematical tools of music theorists:

- Transposition
- Inversion
- The neo-Riemannian *PLR*-group
- Its associated graphs
- An Extension of the *PLR*-group by Fiore-Satyendra.

We will illustrate this extension with an analysis of Hindemith, *Ludus Tonalis*, Fugue in E.

References

Material from this talk is from:

1. Thomas M. Fiore and Ramon Satyendra. Generalized contextual groups. *Music Theory Online*, 11(3), 2005.

2. Alissa Crans, Thomas M. Fiore, and Ramon Satyendra. Musical actions of dihedral groups. *American Mathematical Monthly*, In press since June 2008.

What is Music Theory?

- Music theory supplies us with conceptual categories to organize and understand music.
- David Hume: impressions become tangible and form ideas.
- In other words, music theory provides us with the means to find a good way of hearing a work of music.

Who Needs Music Theory?

- Composers
- Performers
- Listeners

The $\mathbb{Z}_{12}$ Model of Pitch Class

We have a bijection between the set of pitch classes and $\mathbb{Z}_{12}$.

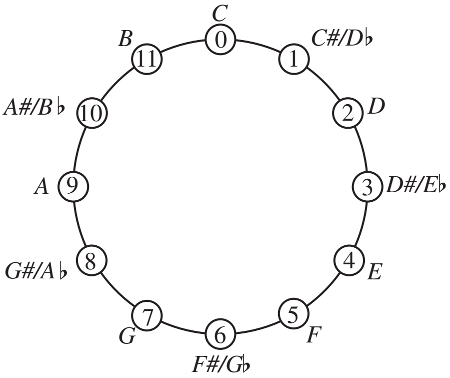


Figure: The musical clock.

Transposition

The bijective function

$$T_n : \mathbb{Z}_{12} \to \mathbb{Z}_{12}$$

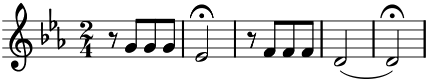
$$T_n(x) := x + n$$

is called *transposition* by musicians, or *translation* by mathematicians.

G	G	G	$E\flat$
7	7	7	3
7	7	7	$T_{-4}(7)$

F	F	F	D
5	5	5	2
5	5	5	$T_{-3}(5)$

$$T_{-4}(7) = 7 - 4 = 3$$



$$T_{-3}(5) = 5 - 3 = 2$$

Inversion

The bijective function

$$I_n : \mathbb{Z}_{12} \to \mathbb{Z}_{12}$$

$$I_n(x) := -x + n$$

is called *inversion* by musicians, or *reflection* by mathematicians.

$$I_0(0) = -0 = 0$$

$$I_0(7) = -7 = 5$$

$\langle C, G \rangle$	$\langle C, F \rangle$
$\langle 0, 7 \rangle$	$\langle I_0(0), I_0(7) \rangle$
$\langle 0, 7 \rangle$	$\langle 0, 5 \rangle$

The T/I-Group

Altogether, these transpositions and inversions form the *T/I-group*.

This is the group of symmetries of the 12-gon, the *dihedral group* of order 24.

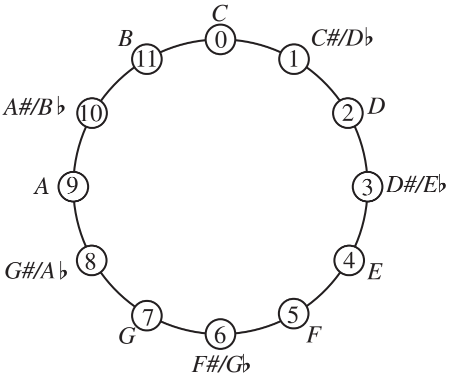


Figure: The musical clock.

Major and Minor Triads

Major and minor triads are very common in Western music.

$$C\text{-major} = \langle C, E, G \rangle = \langle 0, 4, 7 \rangle$$

$$c\text{-minor} = \langle G, E\flat, C \rangle = \langle 7, 3, 0 \rangle$$

The set S of consonant triads	
Major Triads	Minor Triads
$C = \langle 0, 4, 7 \rangle$	$\langle 0, 8, 5 \rangle = f$
$C\sharp = D\flat = \langle 1, 5, 8 \rangle$	$\langle 1, 9, 6 \rangle = f\sharp = g\flat$
$D = \langle 2, 6, 9 \rangle$	$\langle 2, 10, 7 \rangle = g$
$D\sharp = E\flat = \langle 3, 7, 10 \rangle$	$\langle 3, 11, 8 \rangle = g\sharp = a\flat$
$E = \langle 4, 8, 11 \rangle$	$\langle 4, 0, 9 \rangle = a$
$F = \langle 5, 9, 0 \rangle$	$\langle 5, 1, 10 \rangle = a\sharp = b\flat$
$F\sharp = G\flat = \langle 6, 10, 1 \rangle$	$\langle 6, 2, 11 \rangle = b$
$G = \langle 7, 11, 2 \rangle$	$\langle 7, 3, 0 \rangle = c$
$G\sharp = A\flat = \langle 8, 0, 3 \rangle$	$\langle 8, 4, 1 \rangle = c\sharp = d\flat$
$A = \langle 9, 1, 4 \rangle$	$\langle 9, 5, 2 \rangle = d$
$A\sharp = B\flat = \langle 10, 2, 5 \rangle$	$\langle 10, 6, 3 \rangle = d\sharp = e\flat$
$B = \langle 11, 3, 6 \rangle$	$\langle 11, 7, 4 \rangle = e$

Major and Minor Triads

The T/I-group acts on the set S of major and minor triads.

$$T_1\langle 0,4,7\rangle = \langle T_1 0, T_1 4, T_1 7\rangle = \langle 1,5,8\rangle$$

$$I_0\langle 0,4,7\rangle = \langle I_0 0, I_0 4, I_0 7\rangle = \langle 0,8,5\rangle$$

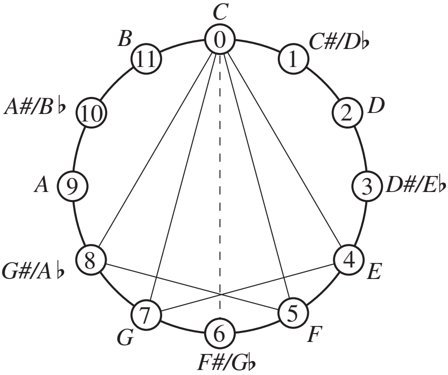


Figure: I_0 applied to a C-major triad yields an f-minor triad.

Neo-Riemannian Music Theory

- Recent work focuses on the neo-Riemannian operations P, L, and R.
- P, L, and R generate a dihedral group, called the *neo-Riemannian group*. As we'll see, this group is *dual* to the T/I group in the sense of Lewin.
- These transformations arose in the work of the 19th century music theorist Hugo Riemann, and have a pictorial description on the Oettingen/Riemann *Tonnetz*.
- P, L, and R are defined in terms of common tone preservation.

The neo-Riemannian Transformation P

We consider three functions

$$P, L, R : S \to S.$$

Let $P(x)$ be that triad of opposite type as x with the first and third notes switched. For example
$P\langle \mathbf{0}, 4, \mathbf{7} \rangle =$
$P(C\text{-major}) =$

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For example

$$P\langle \mathbf{0}, 4, \mathbf{7}\rangle = \langle \mathbf{7}, 3, \mathbf{0}\rangle$$

$$P(C\text{-major}) = c\text{-minor}$$

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$B = \langle 11,3,6\rangle$	$\langle 11,7,4\rangle = e$

The neo-Riemannian Transformations L and R

- Let $L(x)$ be that triad of opposite type as x with the second and third notes switched. For example

$$L\langle 0, \mathbf{4}, \mathbf{7} \rangle = \langle 11, \mathbf{7}, \mathbf{4} \rangle$$

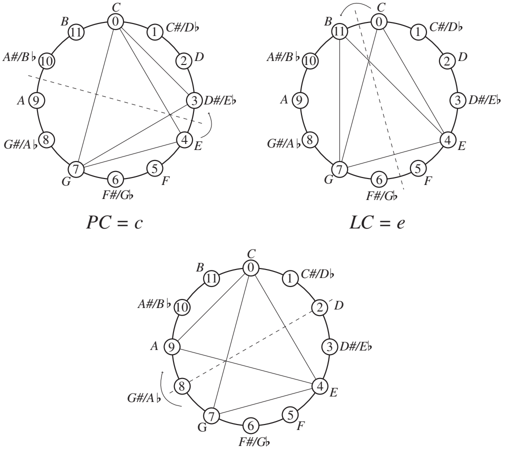
$$L(C\text{-major}) = e\text{-minor}.$$

- Let $R(x)$ be that triad of opposite type as x with the first and second notes switched. For example

$$R\langle \mathbf{0}, \mathbf{4}, 7 \rangle = \langle \mathbf{4}, \mathbf{0}, 9 \rangle$$

$$R(C\text{-major}) = a\text{-minor}.$$

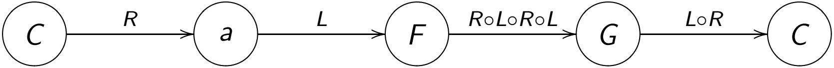
Minimal motion of the moving voice under P, L, and R.



$PC = c$ $LC = e$ $RC = a$

Example: The Elvis Progression

The Elvis Progression I-VI-IV-V-I from 50's Rock is:



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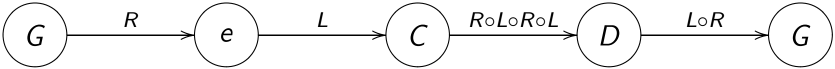
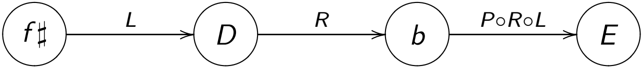


Example: "Oh! Darling" from the Beatles



$E+$ A E
Oh___ Darling please believe me
$f\sharp$ D
I'll never do you no harm
$b7$ $E7$
Be-lieve me when I tell you
$b7$ $E7$ A
I'll never do you no harm

The neo-Riemannian *PLR*-Group and Duality

Definition

The neo-Riemannian PLR-group is the subgroup of permutations of S generated by P, L, and R.

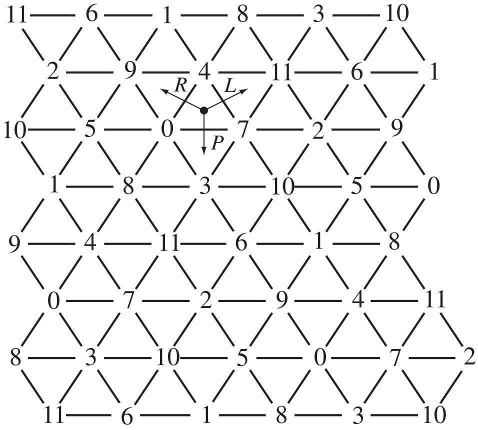
Theorem (Lewin 80's, Hook 2002, ...)

The PLR group is dihedral of order 24 and is generated by L and R.

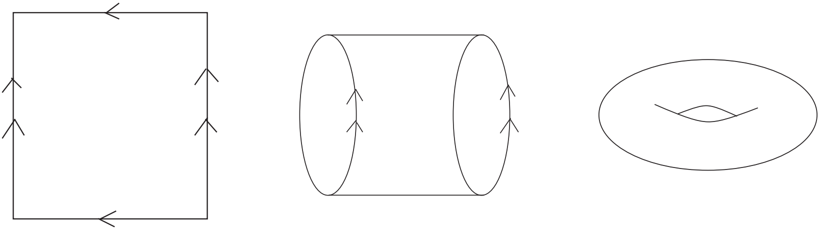
Theorem (Lewin 80's, Hook 2002, ...)

The PLR group is dual to the T/I group in the sense that each is the centralizer of the other in the symmetric group on the set S of major and minor triads. Moreover, both groups act simply transitively on S.

The Oettingen/Riemann *Tonnetz*



The Torus



The Dual Graph to the Tonnetz

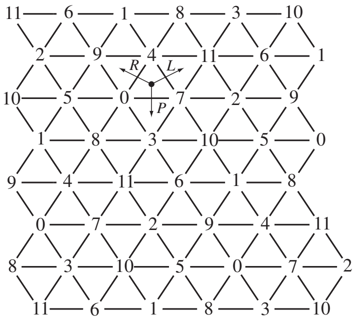


Figure: The *Tonnetz*.

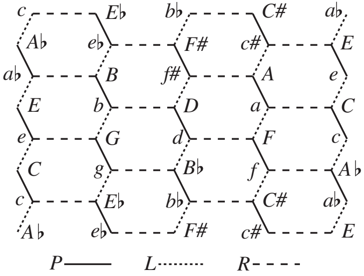


Figure: Douthett and Steinbach's Graph.

The Dual Graph to the Tonnetz

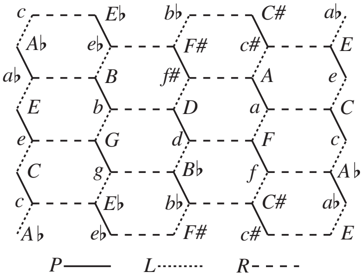


Figure: Douthett and Steinbach's Graph.

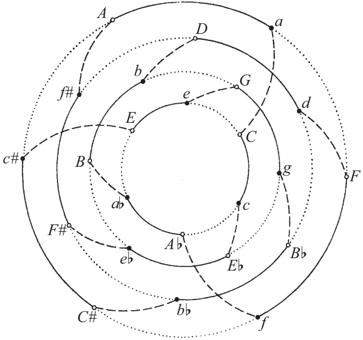
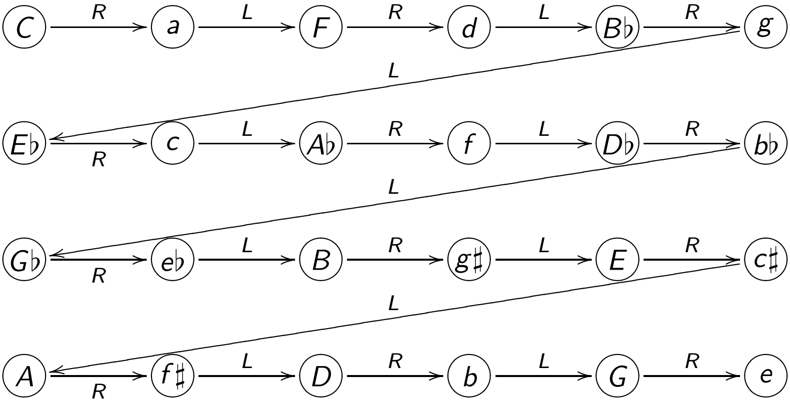


Figure: Waller's Torus.

Beethoven's 9th, 2nd Mvmt, Measures 143-17 (Cohn)



Thus Far We have seen:

- How to encode pitch classes as integers modulo 12, and consonant triads as 3-tuples of integers modulo 12
- How the T/I-group acts componentwise on consonant triads
- How the PLR-group acts on consonant triads
- Duality between the T/I-group and the PLR-group
- Geometric depictions on the torus and musical examples.

But most music does not consist entirely of triads!

Extension of Neo-Riemannian Theory

Theorem (Fiore–Satyendra, 2005)

Let $x_1, \ldots, x_n \in \mathbb{Z}_m$ and suppose that there exist x_q, x_r in the list such that $2(x_q - x_r) \neq 0$. Let S be the family of $2m$ pitch-class segments that are obtained by transposing and inverting the pitch-class segment $X = \langle x_1, \ldots, x_n \rangle$. Then the $2m$ transpositions and inversions act simply transitively on S.

Extension of Neo-Riemannian Theory: Duality

Theorem (Fiore–Satyendra, 2005)

Fix $1 \leq k, \ell \leq n$. *Define*

$$K(Y) := I_{y_k + y_\ell}(Y)$$

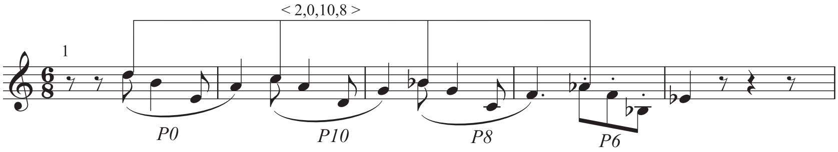
$$Q_i(Y) := \begin{cases} T_i Y & \text{if } Y \text{ is a transposed form of } X \\ T_{-i} Y & \text{if } Y \text{ is an inverted form of } X. \end{cases}$$

Then K *and* Q_1 *generate the centralizer of the* T/I *group of order* $2m$. *This centralizer is called the generalized contextual group. It is dihedral of order* $2m$, *and its centralizer is the mod* m T/I*-group. Moreover, the generalized contextual group acts simply transitively on* S.

Subject of Hindemith, *Ludus Tonalis*, Fugue in *E*

Let's see what this theorem can do for us in an analysis of Hindemith's Fugue in *E* from *Ludus Tonalis*.

Subject:



The Musical Space S in the Fugue in E

Consider the four-note **motive**

$$P_0 = \langle D, B, E, A \rangle$$
$$= \langle 2, 11, 4, 9 \rangle.$$

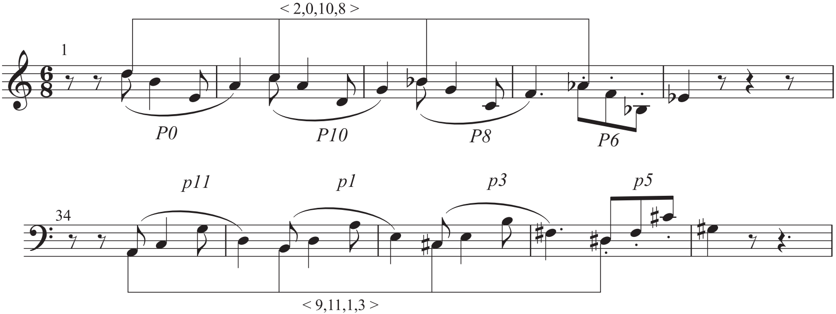
Transposed Forms

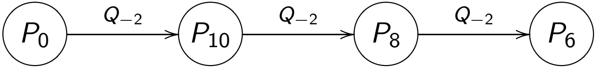
P_0	$\langle 2, 11, 4, 9 \rangle$
P_1	$\langle 3, 0, 5, 10 \rangle$
P_2	$\langle 4, 1, 6, 11 \rangle$
P_3	$\langle 5, 2, 7, 0 \rangle$
P_4	$\langle 6, 3, 8, 1 \rangle$
P_5	$\langle 7, 4, 9, 2 \rangle$
P_6	$\langle 8, 5, 10, 3 \rangle$
P_7	$\langle 9, 6, 11, 4 \rangle$
P_8	$\langle 10, 7, 0, 5 \rangle$
P_9	$\langle 11, 8, 1, 6 \rangle$
P_{10}	$\langle 0, 9, 2, 7 \rangle$
P_{11}	$\langle 1, 10, 3, 8 \rangle$

Inverted Forms

p_0	$\langle 10, 1, 8, 3 \rangle$
p_1	$\langle 11, 2, 9, 4 \rangle$
p_2	$\langle 0, 3, 10, 5 \rangle$
p_3	$\langle 1, 4, 11, 6 \rangle$
p_4	$\langle 2, 5, 0, 7 \rangle$
p_5	$\langle 3, 6, 1, 8 \rangle$
p_6	$\langle 4, 7, 2, 9 \rangle$
p_7	$\langle 5, 8, 3, 10 \rangle$
p_8	$\langle 6, 9, 4, 11 \rangle$
p_9	$\langle 7, 10, 5, 0 \rangle$
p_{10}	$\langle 8, 11, 6, 1 \rangle$
p_{11}	$\langle 9, 0, 7, 2 \rangle$

Q_{-2} Applied to Motive in Subject and I_{11}-Inversion





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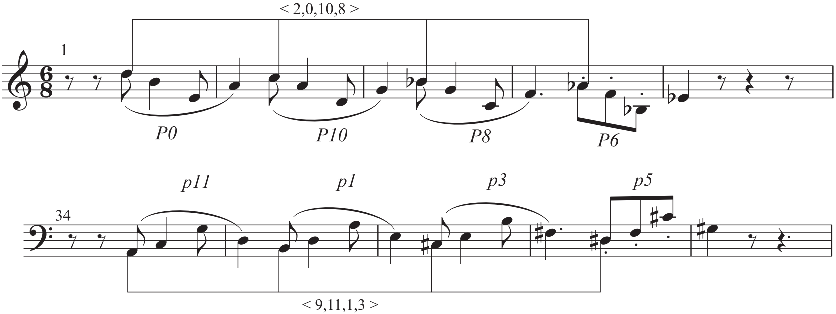
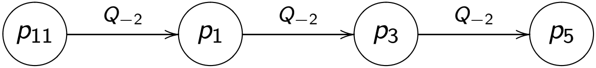
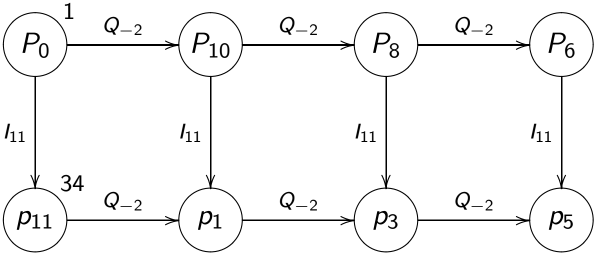


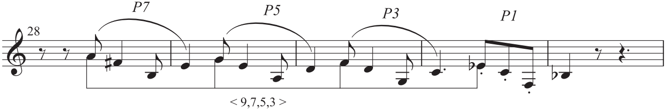


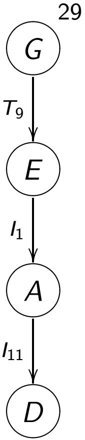
Product Network Encoding Subject and I_{11}-Inversion



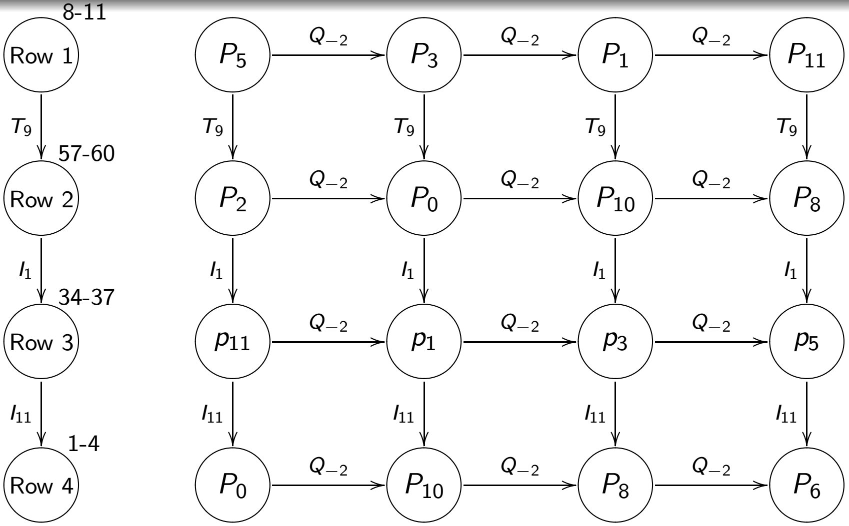
Our Theorem about duality guarantees this diagram commutes!

Self-Similarity: Local Picture





Self-Similarity: Global Picture and Local Picture



Utility of Theorems

- Again, our Theorem about duality allows us to make this product network.
- More importantly, the internal structure of the four-note motive is replicated in transformations that span the work as whole. Thus local and global perspectives are integrated.
- These groups also act on a second musical space S' in the piece, which allows us to see another kind of self-similarity: certain transformational patterns are shared by distinct musical objects!

Summary

- In this lecture I have introduced some of the conceptual categories that music theorists use to make aural impressions into vivacious ideas in the sense of Hume.

- These included: transposition and inversion,
  the *PLR* group,
  its associated graphs on the torus,
  and duality.

Summary

- We have used these tools to find good ways of hearing music from Hindemith, the Beatles, and Beethoven. Some of these ideas would have been impossible without mathematics.
- I hope this introduction to mathematical music theory turned your impressions of music theory into vivacious ideas!