TECHNICAL NOTE: A MINOR CORRECTION OF THEOREM 1.3 FROM [1]

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Abstract. In this short note we correct and simplify the proof of equation (1.8) in Theorem 1.3 of [1]. In the new version (see Theorem 1 below) the noise-to-signal ratio need not be smaller than an absolute constant in order for the stated error guarantee to hold. In the process of proving Theorem 1 we also correct a small (re)normalization issue with Corollary 4 of [2].

We wish to reconstruct a given vector $\mathbf{x}_0 \in \mathbb{C}^n$, up to a global phase factor, from magnitude measurements of the form

$$b_i := |\langle \mathbf{p}_i, \mathbf{x}_0 \rangle|^2 + n_i, \tag{1}$$

where $\mathbf{p}_i \in \mathbb{C}^n$ and $n_i \in \mathbb{R}$ for $i = 1, \ldots, m$. Vectorizing (1) yields

$$\mathbf{b} := |\mathcal{P}\mathbf{x}_0|^2 + \mathbf{n}, \tag{2}$$

where $\mathbf{b}, \mathbf{n} \in \mathbb{R}^m$, $\mathcal{P} \in \mathbb{C}^{m \times n}$, and $|\cdot|^2 : \mathbb{C}^m \to \mathbb{R}^m$ computes the componentwise squared magnitude of each vector entry. We aim to prove the following corrected version of equation (1.8) in Theorem 1.3 from [1] concerning this problem.

Theorem 1. *Let $\mathcal{P} \in \mathbb{C}^{m \times n}$ have its m rows be independently drawn either uniformly at random from the sphere of radius $\sqrt{n}$ in $\mathbb{C}^n$, or else as complex normal random vectors from $\mathcal{N}(0, \mathcal{I}_n/2) + \mathbb{i}\mathcal{N}(0, \mathcal{I}_n/2)$. Then, $\exists$ universal constants $\tilde{B}, \tilde{C}, \tilde{D} \in \mathbb{R}^+$ such that the PhaseLift procedure $\Phi_{\mathcal{P}} : \mathbb{R}^m \to \mathbb{C}^n$ satisfies*

$$\min_{\theta \in [0, 2\pi]} \left\| \Phi_{\mathcal{P}}(\mathbf{b}) - \mathbb{e}^{\mathbb{i}\theta} \mathbf{x}_0 \right\|_2 \leq \tilde{C} \cdot \frac{\|\mathbf{n}\|_1}{m \|\mathbf{x}_0\|_2} \tag{3}$$

for all $\mathbf{x} \in \mathbb{C}^n$ with probability $1 - \mathcal{O}(\mathbb{e}^{-\tilde{B}m})$, provided that $m \geq \tilde{D}n$. Here $\mathbf{b}, \mathbf{n} \in \mathbb{R}^m$ are as in (2).

Our proof relies on a modified version of Corollary 4 from [2]. It reads:

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Corollary 1. *Let* $\mathbf{x}_0 \in \mathbb{C}^n$*, set* $\mathbf{X}_0 = \mathbf{x}_0\mathbf{x}_0^*$*, and let* $\mathbf{X} \succeq 0$ *be such that* $\|\mathbf{X} - \mathbf{X}_0\|_F \leq \eta\|\mathbf{X}_0\|_F = \eta\|\mathbf{x}_0\|_2^2$ *for some* $\eta > 0$*, where* $\|\cdot\|_F$ *denotes the Frobenius norm. Furthermore, let* λ_i *be the* i*-th largest eigenvalue of* $\mathbf{X}$ *and* $\mathbf{v}_i$ *an associated eigenvector, such that the* $\mathbf{v}_i$ *form an orthonormal eigenbasis. Then*

$$\min_{\theta\in[0,2\pi]} \|\mathbb{e}^{\mathbb{i}\theta}\mathbf{x}_0 - \sqrt{\lambda_1}\mathbf{v}_1\|_2 \leq (1+2\sqrt{2})\eta\|\mathbf{x}_0\|_2.$$

See Section 1 for the proof of Corollary 1.

We can now use Corollary 1 to prove Theorem 1:

Proof. Beginning with Equation (1.7) in Theorem 1.3 of [1], we have that

$$\|\mathbf{X} - \mathbf{x}_0\mathbf{x}_0^*\|_F \leq C_0 \cdot \left(\frac{\|\mathbf{n}\|_1}{m\|\mathbf{x}_0\|_2^2}\right)\|\mathbf{x}_0\|_2^2,$$

where $\mathbf{X} \succeq 0$ is the solution to (1.6) in [1], and $C_0 \in \mathbb{R}^+$ is a universal constant. Returning the leading eigenvector of $\mathbf{X}$ reweighed by the square root of its associated eigenvalue now establishes the desired error bound by Corollary 1. □

Please note that no assumptions need to be made concerning the magnitude of the noise vector, $\mathbf{n}$, in Thoerem 1.

1. Proof of Corollary 1

Proof. Note that by construction, the rank one matrix $\mathbf{X}_0$ has one eigenvalue $\nu := \|\mathbf{x}_0\|_2^2$ and all other eigenvalues 0. By Weyl's inequality,

$$\max\{|\nu - \lambda_1|, \lambda_2, \ldots, \lambda_n\} \leq \eta\nu. \tag{4}$$

By orthonormality of the $\mathbf{v}_i$, the spectral norm of the matrix $\mathbf{X} - \nu\mathbf{v}_1\mathbf{v}_1^*$ satisfies

$$\|\mathbf{X} - \nu\mathbf{v}_1\mathbf{v}_1^*\| = \left\|(\lambda_1 - \nu)\mathbf{v}_1\mathbf{v}_1^* + \sum_{j=2}^{n} \lambda_j\mathbf{v}_j\mathbf{v}_j^*\right\| \leq \eta\nu,$$

where the last inequality uses (4). Consequently, by the triangle inequality,

$$\|\mathbf{X}_0 - \nu\mathbf{v}_1\mathbf{v}_1^*\| \leq \|\mathbf{X}_0 - \mathbf{X}\| + \|\mathbf{X} - \nu\mathbf{v}_1\mathbf{v}_1^*\| \leq 2\eta\nu.$$

Thus, we can see that

$$\nu^2 - \nu|\langle\mathbf{x}_0, \mathbf{v}_1\rangle|^2 = \frac{1}{2}\|\mathbf{X}_0 - \nu\mathbf{v}_1\mathbf{v}_1^*\|_F^2 \leq \|\mathbf{X}_0 - \nu\mathbf{v}_1\mathbf{v}_1^*\|^2 \leq 4\eta^2\nu^2, \tag{5}$$

where the second to last inequality follows from the fact that $\mathbf{X}_0 - \nu\mathbf{v}_1\mathbf{v}_1^*$ is at most rank 2.

We next choose $\phi \in [0, 2\pi]$ such that $\langle e^{i\phi}\mathbf{x}_0, \mathbf{v}_1\rangle = |\langle \mathbf{x}_0, \mathbf{v}_1\rangle|$, and then note that

$$
\begin{aligned}
\|e^{i\phi}\mathbf{x}_0 - \sqrt{\nu}\mathbf{v}_1\|_2^2 &= 2\nu - 2\sqrt{\nu}\langle e^{i\phi}\mathbf{x}_0, \mathbf{v}_1\rangle = 2\nu - 2\sqrt{\nu}\cdot|\langle \mathbf{x}_0, \mathbf{v}_1\rangle| \\
&\le \left(2\nu - 2\sqrt{\nu}\cdot|\langle \mathbf{x}_0, \mathbf{v}_1\rangle|\right)\left(\frac{\nu + \sqrt{\nu}\cdot|\langle \mathbf{x}_0, \mathbf{v}_1\rangle|}{\nu}\right) \\
&= \frac{2}{\nu}\left(\nu^2 - \nu|\langle \mathbf{x}_0, \mathbf{v}_1\rangle|^2\right) \le 8\eta^2\nu,
\end{aligned}
\tag{6}
$$

where the last inequality follows from (5). Finally, by the triangle inequality, (4), and (6), we have

$$
\begin{aligned}
\|e^{i\phi}\mathbf{x}_0 - \sqrt{\lambda_1}\mathbf{v}_1\|_2 &\le \|e^{i\phi}\mathbf{x}_0 - \sqrt{\nu}\mathbf{v}_1\|_2 + \|\sqrt{\nu}\mathbf{v}_1 - \sqrt{\lambda_1}\mathbf{v}_1\|_2 \\
&\le 2\sqrt{2}\eta\sqrt{\nu} + \left|\sqrt{\nu} - \sqrt{\lambda_1}\right| \\
&\le 2\sqrt{2}\eta\sqrt{\nu} + \frac{|\nu - \lambda_1|}{\sqrt{\nu} + \sqrt{\lambda_1}} \\
&\le 2\sqrt{2}\eta\sqrt{\nu} + \frac{\eta\nu}{\sqrt{\nu} + \sqrt{\lambda_1}} \\
&\le 2\sqrt{2}\eta\sqrt{\nu} + \eta\sqrt{\nu} = (1 + 2\sqrt{2})\eta\sqrt{\nu}.
\end{aligned}
$$

The desired result now follows.

□

References

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[2] L. Demanet and P. Hand. Stable optimizationless recovery from phaseless linear measurements. *Journal of Fourier Analysis and Applications*, 20(1):199–221, 2014.