A Provably Accurate Algorithm for Recovering Compactly Supported Smooth Functions from Spectrogram Measurements

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***Abstract*—We present an algorithm which is closely related to direct phase retrieval methods that have been shown to work well empirically [1], [2] and prove that it is guaranteed to recover (up to a global phase) a large class of compactly supported smooth functions from their spectrogram measurements. As a result, we take a first step toward developing a new class of practical phaseless imaging algorithms capable of producing provably accurate images of a given sample after it is masked by just a few shifts of a fixed periodic grating.**

***Index Terms*—phase retrieval, phaseless imaging, spectrogram inversion, coded diffraction patterns, Short Time Fourier Transform (STFT) magnitude measurements.**

I. Introduction

Motivated by the plethora of phaseless imaging applications that involve the inversion of spectrogram measurements (see, e.g., [3]), we consider the recovery of a smooth function $f : \mathbb{R} \to \mathbb{C}$ with support contained in $[-\pi, \pi]$ from a finite set of continuous spectrogram measurements of the form

$$Y_{\omega,\ell} := \left| \int_{-\infty}^{\infty} f(x) m \left(x - \frac{2\pi}{L}\ell \right) \mathbb{e}^{-\mathbb{i}x\omega} dx \right|^2. \qquad (1)$$

Here m is a known trigonometric polynomial, and we use d integer frequencies ω and L shifts $\frac{2\pi}{L}\ell$. In this paper, we present an algorithm that will reconstruct f, up to a global phase multiple, by approximating the d lowest frequency Fourier series coefficients of f restricted to $[-\pi, \pi]$.

A. Notation

Let $k \geq 4$. Let d be odd and δ be even with $4\delta \leq d$. Let $\rho \leq \delta$ be even, L divide d, and $L = \rho + \kappa$ for some $2 \leq \kappa \leq \rho$. For n odd, let $[n]_c := \left[\frac{1-n}{2}, \frac{n-1}{2}\right] \cap \mathbb{Z}$ be the set of n consecutive integers centered at the origin, and let

$$\Omega := [d]_c, \quad \mathcal{B} := [d-\delta]_c \quad \text{and} \quad \mathcal{L} := [L]_c.$$

For vectors $\mathbf{x}$ and $\mathbf{y}$, we let $\mathbf{x} \circ \mathbf{y}$ and $\frac{\mathbf{x}}{\mathbf{y}}$ be their componentwise product and quotient, and for $\ell \in \mathbb{Z}$, we let S_ℓ be the circular shift operator defined by $(S_\ell \mathbf{x})_p = x_{p+\ell}$ for $\mathbf{x} = (x_p)_{p\in\Omega}$ (where the addition $p + \ell$ is interpreted to mean the unique element of Ω which is equivalent to $p + \ell$ modulo d). We let $\mathbf{F_d}$ be the $d \times d$ Fourier matrix with entries $(\mathbf{F_d})_{i,j} = \mathbb{e}^{\frac{-2\pi \mathbb{i} ij}{d}}$ for $i, j \in \Omega$, and similarly let $\mathbf{F_L}$ be the $L \times L$ Fourier matrix with indexes in $\mathcal{L}$. We will use C to denote an arbitrary constant which depends only on f and m (and in particular does not depend on d).

B. Main Result

Let $f : \mathbb{R} \to \mathbb{C}$ be a $\mathcal{C}^k$-smooth function, $k \geq 4$, with

$$\text{supp}(f) \subseteq [-\pi, \pi].$$

For $x \in [-\pi, \pi]$, we will write $f(x)$ as its Fourier series

$$f(x) = \sum_{n\in\mathbb{Z}} \widehat{f}(n) \mathbb{e}^{\mathbb{i}nx},$$

where $\widehat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \mathbb{e}^{-\mathbb{i}nx} dx$. We will let

$$D_n := \max_{|n'-n|<\kappa/2} |\widehat{f}(n')|,$$

and assume that $D_n \geq D_{n'}$ whenever $|n| \leq |n'|$.

Remark 1. *Under this assumption, for all $|a| < |n|$, there exists n' such that $|a - n'| < \kappa/2$ and $|\widehat{f}(n')| \geq |\widehat{f}(n)|$.*

Let $m(x)$ be a trigonometric polynomial of the form

$$m(x) = \sum_{p=-\rho/2}^{\rho/2} \widehat{m}(p) \mathbb{e}^{\mathbb{i}px}, \qquad (2)$$

and let $\mathbf{Y} = (Y_{\omega,\ell})_{\omega\in\Omega,\ell\in\mathcal{L}}$ be a $d \times L$ matrix of measurements with entries defined as in (1). The central focus of this paper is Algorithm 1 which allows one to reconstruct the signal $f(x)$ from $\mathbf{Y}$ along with the following theorem guaranteeing its convergence as $d \to \infty$.

Theorem 1. *Let μ be the mask dependent constant defined below in* (6). *If $\mu > 0$, then the output of Algorithm 1, $f_e(x)$, satisfies*

$$\min_{\theta\in[0,2\pi]} \|\mathbb{e}^{\mathbb{i}\theta} f(x) - f_e(x)\|_{L^2(-\pi,\pi)} \leq C \left(\frac{\rho^{1/2}\delta^{1/4}}{\mu^{1/2}} \left(\frac{1}{d}\right)^{(k-3)/2} + \left(\frac{1}{d}\right)^{(k-2)/2} \right).$$

Remark 2. *By imitating the arguments of [8], Proposition 4.1, one may check that it is relatively simple to construct masks such that μ is strictly positive.*

C. Related Work

To the best of our knowledge, Algorithm 1 presented here[1] is the first numerical method theoretically guaranteed to accurately recover a complex-valued function f as above up to a constant phase multiple from STFT magnitude measurements of the form (1). Perhaps the most closely related result to ours is that of Thakur [4] who gives an algorithm for the reconstruction of real-valued bandlimited functions up to a global sign. Gröchenig [5] also considers/surveys similar results in shift-invariant spaces. Other related work includes that of Alaifari et al. [6] which proves (among other things) that one can not hope to stably recover a periodic function up to a single global phase using a trigonometric polynomial mask of degree $\rho/2$ as done below unless its Fourier series coefficients do not vanish on any ρ consecutive integer frequencies in between two other frequencies with nonzero coefficients. This helps motivate the quantity D_n as well as the assumption that $D_n \geq D_{n'}$ whenever $|n| \leq |n'|$. See [7] for similar considerations in the discrete setting.

II. Discretization

Let $P_{\mathcal{B}}f$ be the partial Fourier series

$$P_{\mathcal{B}}f(x) \coloneqq \sum_{n\in\mathcal{B}} \widehat{f}(n)\mathbb{e}^{\mathbb{i}nx},$$

and let $\mathbf{T} \coloneqq (T_{\omega,\ell})_{\omega\in\Omega,\ell\in\mathcal{L}}$ denote the matrix of measurements obtained by replacing f with $P_{\mathcal{B}}f$ in (1), i.e.,

$$T_{\omega,\ell} \coloneqq \left| \int_{-\pi}^{\pi} P_{\mathcal{B}}f(x)m\left(x - \frac{2\pi}{L}\ell\right)\mathbb{e}^{-\mathbb{i}x\omega}dx\right|^2. \quad (3)$$

Our method is based on showing that $\mathbf{Y}$ is well-approximated by $\mathbf{T}$ and by representing $P_{\mathcal{B}}f(x)$ and $m(x)$ with vectors $\mathbf{x} = (x_p)_{p\in\Omega}$ and $\mathbf{y} = (y_p)_{p\in\Omega}$ defined by

$$x_p \coloneqq P_{\mathcal{B}}f\left(\frac{2\pi p}{d}\right) \quad \text{and} \quad y_p \coloneqq m\left(\frac{2\pi p}{d}\right).$$

We will also define $\mathbf{u} = (u_p)_{p\in\Omega}$ and $\mathbf{v} = (v_p)_{p\in\Omega}$ by

$$u_p \coloneqq \widehat{f}(p)\mathbb{1}_{p\in\mathcal{B}} \quad \text{and} \quad v_p \coloneqq \widehat{m}(p)\mathbb{1}_{|p|\leq\rho/2}, \quad (4)$$

where $\mathbb{1}_{p\in\mathcal{B}}$ and $\mathbb{1}_{|p|\leq\rho/2}$ are standard indicator functions. We note that the Fourier transforms of $\mathbf{x}$ and $\mathbf{y}$ satisfy

$$\mathbf{F_d}\mathbf{x} \eqqcolon \widehat{\mathbf{x}} = d\mathbf{u} \quad \text{and} \quad \mathbf{F_d}\mathbf{y} \eqqcolon \widehat{\mathbf{y}} = d\mathbf{v}. \quad (5)$$

Let μ be a mask-dependent constant defined by

$$\mu \coloneqq \inf_{d\geq\rho} \min_{|p|<\kappa, q\in\Omega} |\mathbf{F}_d(\mathbf{v}\circ S_p\overline{\mathbf{v}})_q)| \eqqcolon \inf_{d\in\mathbb{N}} \mu_d. \quad (6)$$

[1]Numerical results available at https://bitbucket.org/charms/blockpr

We note that in light of (5) we have

$$\nu_d \coloneqq \min_{|p|<\kappa, q\in\Omega} \left|\mathbf{F}_d\left(\widehat{\mathbf{y}}\circ S_p\overline{\widehat{\mathbf{y}}})_q\right)\right| = d^2\mu_d \geq d^2\mu. \quad (7)$$

The following lemma shows that the integral in (3) can be replaced by a discrete sum. It is proved by expanding $P_{\mathcal{B}}f$ and m as trigonometric polynomials and using the fact that

$$2\pi\sum_{p\in\Omega}\mathbb{e}^{2\pi\mathbb{i}pj/d} = d\int_{-\pi}^{\pi} e^{\mathbb{i}jx}dx \quad \forall j\in\Omega.$$

Lemma 1. *Let $\ell\in\mathcal{L}$, $\omega\in\Omega$, and let $\widetilde{\ell} = \frac{2\pi\ell}{L}$. Then,*

$$\int_{-\pi}^{\pi} P_{\mathcal{B}}f(x)m\left(x-\widetilde{\ell}\right)\mathbb{e}^{-\mathbb{i}x\omega}dx = \frac{2\pi}{d}\sum_{p\in\Omega} x_p y_{p-\ell\frac{d}{L}}\mathbb{e}^{-2\pi\mathbb{i}\omega p/d}.$$

We may use Lemma 1 to prove the following result which shows that $\mathbf{T}$ converges to $\mathbf{Y}$ as $d\to\infty$.

Lemma 2. *Let $\mathbf{E} \coloneqq \mathbf{Y} - \mathbf{T}$. Then*

$$\|\mathbf{E}\|_\infty \leq C\rho\left(\frac{1}{d}\right)^k, \quad \textit{and} \quad (8)$$

$$E_{\omega,\ell} = 0 \quad \textit{whenever } |\omega| \leq \frac{d-1}{2} - \delta. \quad (9)$$

Proof. For $\omega\in\Omega$ and $\ell\in\mathcal{L}$, let

$$M_{\omega,\ell} \coloneqq \int_{-\pi}^{\pi} f(x)m\left(x - \frac{2\pi}{d}\ell\right)\mathbb{e}^{-\mathbb{i}x\omega}dx, \quad \text{and}$$

$$U_{\omega,\ell} \coloneqq \int_{-\pi}^{\pi} P_{\mathcal{B}}f(x)m\left(x - \frac{2\pi}{d}\ell\right)\mathbb{e}^{-\mathbb{i}x\omega}dx.$$

It suffices to show that

$$|U_{\omega,\ell}| \leq C \quad (10)$$

$$\text{and} \quad |E'_{\omega,\ell}| \leq C\rho\left(\frac{1}{d}\right)^k. \quad (11)$$

Then, letting $E'_{\omega,\ell} \coloneqq M_{\omega,\ell} - U_{\omega,\ell}$, we will have

$$\begin{aligned} |E_{\omega,\ell}| &= ||M_{\omega,l}|^2 - |U_{\omega,l}|^2| \\ &= (|M_{\omega,l}| + |U_{\omega,l}|)||M_{\omega,l}| - |T_{\omega,l}|| \\ &\leq (2|U_{\omega,l}| + |E'_{\omega,l}|)|E'_{\omega,l}| \\ &\leq C\left(1+\rho\left(\frac{1}{d}\right)^k\right)\rho\left(\frac{1}{d}\right)^k \\ &\leq C\rho\left(\frac{1}{d}\right)^k. \end{aligned}$$

Since $m(x)$ is a trigonometric polynomial, we see

$$\|\mathbf{y}\|_\infty \leq \|m\|_\infty \leq C,$$

and since f is $\mathcal{C}^2$-smooth and compactly supported, we have

$$\|\mathbf{x}\|_\infty \leq \|P_{\mathcal{B}}f\|_\infty \leq \sum_{n\in\mathbb{Z}} |\widehat{f}(n)| \leq C.$$

Therefore, using Lemma 1, we see that

$$|U_{\omega,l}| = \left|\frac{2\pi}{d}\sum_{p\in\Omega} x_p y_{p-\ell\frac{d}{L}}\mathbb{e}^{-2\pi\mathbb{i}\omega p/d}\right| \leq C, \quad (12)$$

and so (10) follows. To prove (11), we note that

$$f(x) - P_{\mathcal{B}} f(x) = \sum_{n \notin \mathcal{B}} \widehat{f}(n) \mathrm{e}^{\mathrm{i} n x},$$

and therefore

$$\begin{aligned} E'_{\omega,l} &= \int_{-\pi}^{\pi} (f(x) - P_{\mathcal{B}} f(x))\, m\left(x - \frac{2\pi\ell}{L}\right) \mathrm{e}^{-\mathrm{i}\omega x} dx \\ &= \sum_{n \notin \mathcal{B}} \sum_{p=-\rho/2}^{\rho/2} \widehat{f}(n) \widehat{m}(p) \mathrm{e}^{-\frac{2\pi \mathrm{i} p \ell}{L}} \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}(n+p-\omega)x} dx. \end{aligned}$$

The inner integral is zero unless $\omega = n + p$. Therefore,

$$\begin{aligned} |E'_{\omega,l}| &\leq 2\pi \sum_{n \notin \mathcal{B}, |\omega - n| \leq \rho/2} \left|\widehat{f}(n)\right| |\widehat{m}(\omega - n)| \\ &\leq 2\pi \sup_{n \notin \mathcal{B}} \left\{|\widehat{f}(n)|\right\} \rho \sup_{|n| \leq \rho/2} |\widehat{m}(n)| \\ &\leq C\rho \left(\frac{1}{d}\right)^k, \end{aligned}$$

where we used the facts $\widehat{f}(n) = \mathcal{O}(n^{-k})$ and that $n > \frac{d}{4}$ for all $n \notin \mathcal{B}$. This proves (8). Equation (9) follows from noting that the condition $\omega = n + p$ can never hold when $n \notin \mathcal{B}$, $|\omega| \leq \frac{d-1}{2} - \delta$ and $|p| \leq \rho/2$. ☐

III. Wigner Deconvolution

In this section, we apply a discrete, aliased Wigner deconvolution approach, similar to Section 3 of [8], to solve for a portion of the Fourier autocorrelation matrix $\widehat{\mathbf{x}}\widehat{\mathbf{x}}^*$. It follows from (12) that

$$T_{\omega,\ell} = \frac{4\pi^2}{d^2} \left| \sum_{p \in \Omega} x_p y_{p - \ell\frac{d}{L}} \mathrm{e}^{-2\pi \mathrm{i} \omega p / d} \right|^2.$$

Up to a scaling factor of $4\pi^2/d^2$, these measurements coincide with the measurements considered in [8].

Let $\widetilde{\mathbf{T}} := \mathbf{F_L} \mathbf{T}^T \mathbf{F_d}^T$, and let $\widetilde{\mathbf{E}} := \mathbf{F_L} \mathbf{E}^T \mathbf{F_d}^T$. Since $\frac{1}{\sqrt{d}} \mathbf{F_d}$ and $\frac{1}{\sqrt{L}} \mathbf{F_L}$ are unitary, we may use Lemma 2 to see

$$\|\widetilde{\mathbf{E}}\|_F \leq \sqrt{dL} \|\mathbf{E}\|_F \leq \sqrt{2d\delta} L \|\mathbf{E}\|_\infty \leq CL\rho\delta^{1/2} \left(\frac{1}{d}\right)^{k-1/2}. \tag{13}$$

It follows from Theorem 4, Equation 3.2, of [8] that

$$\widetilde{T}_{\ell,\omega} - \widetilde{E}_{\ell,\omega} = \frac{4\pi^2 L}{d^4} \sum_{p \in \left[\frac{d}{L}\right]_c} \left(\mathbf{F_d}\left(\widehat{\mathbf{x}} \circ S_{pL-\ell} \overline{\widehat{\mathbf{x}}}\right)\right)_\omega \left(\mathbf{F_d}\left(\widehat{\mathbf{y}} \circ S_{\ell - pL} \overline{\widehat{\mathbf{y}}}\right)\right)_\omega,$$

where, as in Section I, $(S_\ell \mathbf{x})_p = \mathbf{x}_{p+\ell}$ for all $\ell \in \mathbb{Z}$. By construction, we have that $\mathrm{supp}(\widehat{\mathbf{y}}) \subseteq [\rho + 1]_c$. Therefore, if $1 - \kappa \leq \ell \leq \kappa - 1$, we may use the same reasoning as in the proof of Lemma 10 of [8], to see that $\widehat{\mathbf{y}} \circ S_{\ell - pL} \overline{\widehat{\mathbf{y}}} = 0$ except for when $p = 0$. Therefore,

$$\widetilde{T}_{\ell,\omega} - \widetilde{E}_{\ell,\omega} = \frac{4\pi^2 L}{d^4} \left(\mathbf{F_d}\left(\widehat{\mathbf{x}} \circ S_{-\ell} \overline{\widehat{\mathbf{x}}}\right)\right)_\omega \left(\mathbf{F_d}\left(\widehat{\mathbf{y}} \circ S_{\ell} \overline{\widehat{\mathbf{y}}}\right)\right)_\omega.$$

Changing variables $\ell \to -\ell$ we see that

$$\left(\mathbf{F_d}\left(\widehat{\mathbf{x}} \circ S_\ell \overline{\widehat{\mathbf{x}}}\right)\right)_\omega = \frac{d^4}{4\pi^2 L} \left(\frac{\widetilde{T}_{-\ell,\omega} - \widetilde{E}_{-\ell,\omega}}{(\mathbf{F_d}(\widehat{\mathbf{y}} \circ S_{-\ell} \overline{\widehat{\mathbf{y}}}))_\omega}\right),$$

and so

$$\widehat{\mathbf{x}} \circ S_\ell \overline{\widehat{\mathbf{x}}} = \frac{d^4}{4\pi^2 L} \mathbf{F_d}^{-1} \left(\frac{\widetilde{\mathbf{T}}_{-\ell} - \widetilde{\mathbf{E}}_{-\ell}}{(\mathbf{F_d}(\widehat{\mathbf{y}} \circ S_{-\ell} \overline{\widehat{\mathbf{y}}}))}\right), \tag{14}$$

where $\mathbf{M}_j$ denotes the j-th column of a matrix $\mathbf{M}$ and we define vector division componentwise.

Let $T_\kappa : \mathbb{C}^{d \times d} \to \mathbb{C}^{d \times d}$ be the restriction operator

$$T_\kappa(\mathbf{M})_{ij} = \begin{cases} M_{i,j} & \text{if } |i - j| < \kappa \\ 0 & \text{otherwise} \end{cases}.$$

We may rewrite (14) in matrix form as

$$T_\kappa(\widehat{\mathbf{x}}\widehat{\mathbf{x}}^*) = \mathbf{X} + \mathbf{N}, \tag{15}$$

where the matrix $\mathbf{X} = (X_{i,j})_{i,j \in \Omega}$ has entries defined by

$$X_{i,j} = \begin{cases} \frac{d^4}{4\pi^2 L} \left(\mathbf{F_d}^{-1} \left(\frac{\widetilde{\mathbf{T}}_{i-j}}{(\mathbf{F_d}(\widehat{\mathbf{y}} \circ S_{i-j} \overline{\widehat{\mathbf{y}}}))}\right)\right)_i & \text{if } |i - j| < \kappa \\ 0 & \text{otherwise} \end{cases}, \tag{16}$$

and $\mathbf{N}$ is defined similarly using $\widetilde{\mathbf{E}}$ in place of $\widetilde{\mathbf{T}}$. Let $\mathbf{R} = (R_{i,j})_{i \in \Omega, j \in [2\kappa - 1]_c}$ be the $d \times (2\kappa - 1)$ matrix with entries $R_{i,j} = N_{i,i+j}$ so that the columns of $\mathbf{R}$ are the diagonal bands of $\mathbf{N}$ within κ of the main diagonal. By (7), we may bound the ℓ^2-norm of each column of $\mathbf{R}$ by

$$\begin{aligned} \|\mathbf{R}_j\|_2 &= \left\| \frac{d^4}{4\pi^2 L} \mathbf{F_d}^{-1} \left(\frac{\widetilde{\mathbf{E}}_{-j}}{(\mathbf{F_d}(\widehat{\mathbf{y}} \circ S_{-j} \overline{\widehat{\mathbf{y}}}))}\right) \right\|_2 \\ &\leq \frac{d^{7/2}}{4\pi^2 L \nu_d} \|\widetilde{\mathbf{E}}_{-j}\|_2 \leq \frac{d^{3/2}}{4\pi^2 L \mu} \|\widetilde{\mathbf{E}}_{-j}\|_2. \end{aligned}$$

Since $\mathbf{N}$ is a banded matrix, (13) implies

$$\|\mathbf{N}\|_F = \|\mathbf{R}\|_F \leq \frac{d^{3/2}}{4\pi^2 L \mu} \|\widetilde{\mathbf{E}}\|_F \leq C \frac{\rho \delta^{1/2}}{\mu} \left(\frac{1}{d}\right)^{k-2}. \tag{17}$$

Dividing both sides of (15) by d^2, using (5), and applying the Hermitianizing operator $H(\mathbf{M}) = \frac{1}{2}(\mathbf{M} + \mathbf{M}^*)$ yields

$$T_\kappa(\mathbf{u}\mathbf{u}^*) = \mathbf{A} + \widetilde{\mathbf{N}}. \tag{18}$$

where $\mathbf{A} = d^{-2} H(\mathbf{X})$, and $\widetilde{\boldsymbol{N}} := d^{-2} H(\mathbf{N})$. By (17), and the triangle inequality, we have

$$\|\widetilde{\boldsymbol{N}}\|_F \leq C \frac{\rho \delta^{1/2}}{\mu} \left(\frac{1}{d}\right)^k. \tag{19}$$

IV. Angular Synchronization

In this section, we will use a greedy angular synchronization approach to recover the Fourier coefficients of f. For each $n \in \mathcal{B}$, the greedy algorithm, Algorithm 2, outputs a sequence $\{n_\ell\}_{\ell=0}^{b}$ where $n_0 = \arg\max_{n\in\mathcal{B}} a_n$ and $n_b = n$. Given that sequence, we let

$$\alpha_n := \sum_{l=0}^{b-1} \arg\left(A_{n_{\ell+1},n_\ell}\right).$$

To understand this definition, let

$$\theta_0 := \arg(\widehat{f}(n_0)) \quad \text{and} \quad \tau_n := \sum_{l=0}^{b-1} \arg\left((\mathbf{u}\mathbf{u}^*)_{n_{\ell+1},n_\ell}\right).$$

Then, we have $\tau_n = \arg\left(\widehat{f}(n)\right) - \theta_0$, and therefore

$$e^{-\mathrm{i}\theta_0}\widehat{f}(n) = |\widehat{f}(n)|e^{i\tau_n}$$

for all $n \in \mathcal{B}$. (Note that n_0 does not depend on n.) Since $\mathbf{A}$ is a noisy approximation of $\mathbf{u}\mathbf{u}^*$, we intuitively view α_n as a noisy approximation of τ_n (up to a phase shift θ_0). Lemma 3 will show that this intuition is correct when $|\widehat{f}(n)|$ is sufficiently large. Due to [9, Lemma 3], for all $n \in \mathcal{B}$ we have

$$\left|\sqrt{|A_{n,n}|} - |\widehat{f}(n)|\right|^2 \leq 3\|\widetilde{\boldsymbol{N}}\|_\infty. \tag{20}$$

Therefore, we set $a_n := \sqrt{|A_{n,n}|}$ and define the output of Algorithm 1 to be the trigonometric polynomial

$$f_e(x) := \sum_{n\in\mathcal{B}} a_n e^{\mathrm{i}\alpha_n} e^{\mathrm{i}nx}.$$

The following lemma shows that α_n is indeed a good approximation of τ_n when $|\widehat{f}(n)|$ is sufficiently large. Its proof is nearly identical to the proof of [9, Lemma 4], but uses Lemma 4 stated below in place of the "flat vector" condition considered there.

Lemma 3. *Let L_f be the set*

$$L_f = \{n \in \mathcal{B} : |\widehat{f}(n)|^2 \geq 48\|\widetilde{\boldsymbol{N}}\|_\infty\}.$$

Then, for all $n \in L_f$

$$|e^{\mathrm{i}\tau_n} - e^{\mathrm{i}\alpha_n}| \leq \frac{2\pi d\|\widetilde{\boldsymbol{N}}\|_\infty}{|\widehat{f}(n)|^2}.$$

As mentioned above, the key to modifying the proof of [9, Lemma 4] in order to prove Lemma 3 is the following lemma, which shows that Algorithm 2 will only select entries n_ℓ corresponding to large Fourier coefficients.

Lemma 4. *Let $n \in L_f$, and let $\{n_\ell\}_{\ell=0}^{b}$ be the sequence output by Algorithm 2. Then,*

$$|\widehat{f}(n_\ell)| \geq \frac{|\widehat{f}(n)|}{2} \quad \textit{for all } 0 \leq \ell \leq b.$$

Proof. When $\ell = b$, the claim is immediate. For $0 \leq \ell \leq b-1$, we have $a_{n_\ell} = \max_{m\in I_\ell} a_m$ for some interval I_ℓ of length 2κ, which is centered at some $|a| \leq |n|$. Therefore, letting $\epsilon = \sqrt{3\|\widetilde{\boldsymbol{N}}\|_\infty}$, we see that by (20) and Remark 1,

$$|\widehat{f}(n_\ell)| \geq \max_{m\in I_\ell} a_m - \epsilon \geq \max_{m\in I_\ell} |\widehat{f}(m)| - 2\epsilon \geq |\widehat{f}(n)| - 2\epsilon.$$

The result follows by noting that $\epsilon < \frac{|\widehat{f}(n)|}{4}$ for $n \in L_f$. □

Together, (20) and Lemma 3 allow us to prove the following lemma showing that $f_e(x)$ approximates $P_\mathcal{B} f(x)$.

Lemma 5. *The output of Algorithm 1 satisfies*

$$\left\|e^{-\mathrm{i}\theta_0}P_\mathcal{B} f(x) - f_e(x)\right\|_{L^2(-\pi,\pi)} \leq C\, d^{\frac{3}{2}}\sqrt{\|\widetilde{\boldsymbol{N}}\|_\infty}.$$

Proof. Recall the vector $\mathbf{u}$ defined in (4) and, for $n \in \Omega$, let

$$u_n' := a_n e^{\mathrm{i}\alpha_n} \quad \text{and} \quad u_n'' := |u_n| e^{\mathrm{i}\alpha_n}.$$

By construction, for all $n \notin \mathcal{B}$ we have $a_n = u_n = 0$. Therefore the supports of $\mathbf{u}' := (u_n')_{n\in\Omega}$ and $\mathbf{u}'' := (u_n'')_{n\in\Omega}$ are contained in $\mathcal{B}$. By Parseval's identity, we see

$$\begin{aligned}
&\left\|e^{-\mathrm{i}\theta_0}P_\mathcal{B} f(x) - \sum_{n\in\mathcal{B}} a_n e^{\mathrm{i}\alpha_n}e^{\mathrm{i}nx}\right\|_{L^2(-\pi,\pi)} \\
=&\left\|e^{-\mathrm{i}\theta_0}\sum_{n\in\mathcal{B}} u_n e^{\mathrm{i}nx} - \sum_{n\in\mathcal{B}} u_n' e^{\mathrm{i}nx}\right\|_{L^2(-\pi,\pi)} \\
\leq&\sqrt{2\pi}\left\|e^{-\mathrm{i}\theta_0}\mathbf{u} - \mathbf{u}'\right\|_{\ell_2} \\
\leq&\sqrt{2\pi}\left\|e^{-\mathrm{i}\theta_0}\mathbf{u} - \mathbf{u}''\right\|_{\ell_2} + \sqrt{2\pi}\|\mathbf{u}'' - \mathbf{u}'\|_{\ell_2} \\
=:& I_1 + I_2.
\end{aligned}$$

Using Lemma 3 and the fact that $|e^{\mathrm{i}\tau_n} - e^{\mathrm{i}\alpha_n}| \leq 2$, we have

$$\begin{aligned}
I_1^2 &= 2\pi\sum_{n\in\mathcal{B}} |u_n|^2|e^{\mathrm{i}\tau_n} - e^{\mathrm{i}\alpha_n}|^2 \\
&\leq C\sum_{n\in\mathcal{B}\setminus L_f} |u_n|^2 + C\sum_{n\in L_f} d^2\,\|\widetilde{\boldsymbol{N}}\|_\infty^2\,|\widehat{f}(n)|^{-2} \\
&\leq C\,d\,\|\widetilde{\boldsymbol{N}}\|_\infty + C\sum_{n\in L_f} d^2\|\widetilde{\boldsymbol{N}}\|_\infty \\
&\leq C\,d^3\|\widetilde{\boldsymbol{N}}\|_\infty.
\end{aligned}$$

To estimate I_2, we recall (20) and note

$$I_2^2 = 2\pi\sum_{n\in\mathcal{B}}\left||u_n| - a_n\right|^2 \leq Cd\|\widetilde{\boldsymbol{N}}\|_\infty. \qquad \square$$

We now use Lemma 5 as well as the uniform convergence of the partial Fourier series $P_\mathcal{B} f$ to prove Theorem 1.

Proof. [The Proof of Theorem 1] By the triangle inequality,

$$\begin{aligned}
&\min_{\theta\in[0,2\pi]}\left\|e^{\mathrm{i}\theta}f(x) - \sum_{n\in\mathcal{B}} a_n e^{\mathrm{i}\alpha_n}e^{\mathrm{i}nx}\right\|_{L^2(-\pi,\pi)} \\
&\leq \|f - P_\mathcal{B} f\|_{L^2(-\pi,\pi)} \\
&\qquad + \left\|e^{-\mathrm{i}\theta_0}P_\mathcal{B} f(x) - \sum_{n\in\mathcal{B}} a_n e^{\mathrm{i}\alpha_n}e^{\mathrm{i}nx}\right\|_{L^2(-\pi,\pi)},
\end{aligned}$$

where $\theta_0 = \arg(\widehat{f}(n_0))$. By (19) and Lemma 5, we have

$$\left\|e^{-\mathrm{i}\theta_0}P_\mathcal{B} f(x) - f_e(x)\right\|_2 \leq C\frac{\rho^{1/2}\delta^{1/4}}{\mu^{1/2}}\left(\frac{1}{d}\right)^{(k-3)/2}.$$

To bound the first term we see that by Parseval's identity, the fact that $d > 4\delta$, and the fact that $|\widehat{f}(n)| = \mathcal{O}(d^{-k})$

$$\|f - P_{\mathcal{B}}f\|_2^2 = 2\pi \sum_{n \notin B} |\widehat{f}(n)|^2 \le C \sum_{n \ge \frac{d}{4}} \frac{1}{n^{2k}} \le C \frac{1}{d^{2k-1}}. \square$$

Algorithm 1 Wigner Deconvolution and Angular Synchronization for Bandlimited Masks

Inputs

1) Matrix $\mathbf{Y} = (Y_{\omega,\ell})_{\omega \in \Omega, \ell \in \mathcal{L}}$ of spectogram measurements defined as in (1).
2) Trigonometric polynomial mask of the form (2).

Steps

1) Define vector $\mathbf{y} = (y_p)_{p \in \Omega}$ by $y_p = m\left(\frac{2\pi p}{d}\right)$.
2) Let $\kappa = L - \rho$, and for $1 - \kappa \le \ell \le \kappa - 1$ estimate
$$\mathbf{F_d}\left(\widehat{\mathbf{x}} \circ S_\ell \overline{\widehat{\mathbf{x}}}\right) \approx \frac{d^4}{4\pi^2 L} \left(\frac{(\mathbf{F_L} \mathbf{Y}^T \mathbf{F_d}^T)_{-\ell}}{(\mathbf{F_d}(\widehat{\mathbf{y}} \circ S_{-\ell} \overline{\widehat{\mathbf{y}}}))} \right).$$
3) Invert the Fourier transforms above to recover estimates of the vectors $\widehat{\mathbf{x}} \circ S_\ell \overline{\widehat{\mathbf{x}}}$.
4) Organize these vectors into a banded matrix, $\mathbf{X}$ described as in (16).
5) Hermitianize $\mathbf{X}$ and divide by d^2 to obtain the matrix $\mathbf{A} = (A_{i,j})_{i,j \in \Omega}$ as described in (18).
6) Estimate $|\widehat{f}(n)| \approx a_n = \sqrt{|A_{n,n}|}$.
7) For $n \in \mathcal{B}$, choose $\{n_\ell\}_{\ell=0}^{b}$ according to Algorithm 2.
8) Approximate
$$\arg\left(\widehat{f}(n)\right) \approx \alpha_n = \sum_{\ell=0}^{b-1} \arg\left(A_{n_{\ell+1}, n_\ell}\right).$$

Output

An approximation of f given by
$$f_e(x) = \sum_{n \in \mathcal{B}} a_n \mathbb{e}^{\mathbb{i}\alpha_n} \mathbb{e}^{\mathbb{i}nx}.$$
.

Algorithm 2 Greedy Entry Selection

Inputs

1) Vector of amplitudes $\mathbf{a} = (a_n)_{n \in \Omega}, \quad a_n = \sqrt{|A_{n,n}|}$.
2) Entry $n \in \mathcal{B}$.

Steps

1) Choose $n_0 = \arg\max_{n \in \mathcal{B}} a_n$.
2) Let $b = 0$.
3) While: $|n - n_b| > \kappa/2$
4) If: $n > n_b$, let $n_{b+1} \leftarrow \arg\max_{n_b < m < n_b + \kappa} a_m$
5) If: $n < n_b$, let $n_{b+1} \leftarrow \arg\max_{n_b - \kappa < m < n_b} a_m$
6) $b \leftarrow b + 1$
7) $n_b \leftarrow n$

Output

A sequence $\{n_\ell\}_{\ell=0}^{b}$, $|n_{\ell+1} - n_\ell| < \kappa/2$, $n_b = n$, $b \le d$.

V. Future Work

The work here shows that, under suitable regularity assumptions, we may recover a continuous signal $f(x)$ from a $d \times L$ matrix of phaseless measurements. We believe that this paper naturally opens up several research directions for future work. Firstly, one might replace the assumption that $m(x)$ is a trigonometric polynomial with the assumption that $m(x)$ is compactly supported in space. This would lead to a measurement setup closely related to ptychographic imaging. Also, in [8], it is shown that in the discrete setting, a discrete Wigner deconvolution approach can be applied to a $K \times L$ measurement matrix for some $K < d$ and that this approach is robust to additive noise. It is likely that analogous techniques can be applied in the continuous setting when the matrix $\mathbf{Y}$ is subsampled in frequency and corrupted by additive noise. Lastly, one might also extend these results to functions of two variables $f(x, y)$.

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References

[1] N. Sissouno, F. Boßmann, F. Filbir, M. Iwen, M. Kahnt, R. Saab, C. Schroer, and W. zu Castell, "A direct solver for the phase retrieval problem in ptychographic imaging," *Mathematics and Computers in Simulation*, 2019.

[2] S. Merhi, A. Viswanathan, and M. Iwen, "Recovery of compactly supported functions from spectrogram measurements via lifting," *Proc. of Sampling Theory and Applications (SampTA)*, pp. 538–542, 2017.

[3] J. M. Rodenburg, "Ptychography and related diffractive imaging methods," *Advances in imaging and electron physics*, vol. 150, pp. 87–184, 2008.

[4] G. Thakur, "Reconstruction of Bandlimited Functions from Unsigned Samples," *J Fourier Anal Appl*, vol. 17, no. 4, pp. 720 – 732, 2011.

[5] K. Gröchenig, "Phase-Retrieval in Shift-Invariant Spaces with Gaussian Generator," *Preprint, arXiv:1911.11050*, 2019.

[6] R. Alaifari, I. Daubechies, P. Grohs, and R. Yin, "Stable phase retrieval in infinite dimensions," *Foundations of Computational Mathematics*, vol. 19, no. 4, pp. 869–900, 2019.

[7] M. A. Iwen, S. Merhi, and M. Perlmutter, "Lower Lipschitz bounds for phase retrieval from locally supported measurements," *Applied and Computational Harmonic Analysis*, vol. 47, no. 2, pp. 526 – 538, 2019.

[8] M. Perlmutter, S. Merhi, A. Viswanathan, and M. Iwen, "Inverting spectrogram measurements via aliased Wigner distribution deconvolution and angular synchronization," *Preprint, arXiv:1907.10773*, 2019.

[9] M. A. Iwen, A. Viswanathan, and Y. Wang, "Fast Phase Retrieval from Local Correlation Measurements," *SIAM J. on Imaging Sciences*, vol. 9, no. 4, pp. 1655–1688, 2016.