Direct Methods for Reconstruction of Functions and their Edges from Non-Uniform Fourier Data

Aditya Viswanathan
aditya@math.msu.edu

ICERM Research Cluster
Computational Challenges in Sparse and Redundant Representations
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Outline

1. Introduction

2. Non-Uniform Fourier Reconstruction
   - Uniform Re-Sampling
   - Convolutional Gridding
   - Non-Uniform FFTs

3. Edge Detection
   - Concentration Method

4. Spectral Re-Projection
Model Problem

Let $f$ be defined in $\mathbb{R}$ with support in $[-\pi, \pi)$. Given

$$\hat{f}(\omega_k) = \langle f, e^{i\omega_k x} \rangle, \quad k = -N, \ldots, N,$$

($\omega_k$ not necessarily $\in \mathbb{Z}$)

compute

• an approximation to the underlying function $f$,

• an approximation to the locations and values of jumps in the underlying function; i.e.,

$$[f](x) := f(x^+) - f(x^-).$$
Motivating Application – Magnetic Resonance Imaging

Physics of MRI dictates that the MR scanner collect samples of the Fourier transform of the specimen being imaged.
Collecting non-uniform measurements has certain advantages; for example, they are easier and faster to collect, and, aliased images retain diagnostic qualities.
Challenges in Non-Uniform Reconstruction

• Computational Issues

  • The FFT is not directly applicable.

  • Direct versus iterative solvers . . .

• Sampling Issues
  Typical MR sampling patterns have non uniform sampling density; i.e., the high modes are sparsely sampled ($|\omega_k - k| > 1$ for $k$ large).

• Other Issues
  Piecewise-smooth functions and Gibbs artifacts
Why Direct Methods?

• Faster (by a small but non-negligible factor) than iterative formulations.

• Provide good initial solutions to seed iterative algorithms.

• Sometimes used as preconditioners in solving iterative formulations.
Jittered Sampling: $\omega_k = k \pm U[0, \mu], \quad k = -N, \cdots, N$

$U[a, b]$: iid uniform distribution in $[a, b]$ with equiprobable $+/-$ jitter.
Log Sampling: \[ \omega_{k+} = a e^{b(2\pi k)}, \quad k = 1, \ldots, N, \quad b = \frac{\ln(N/a)}{2\pi N} \]

\(a\) controls the closest sampling point to the origin.
Model (1D) Sampling Patterns

Polynomial Sampling: 
\[ \omega_{k+} = a k^b, \quad k = 1, \ldots, N, \quad a = \frac{1}{N^{b-1}} \]

\( b \) is the polynomial order.
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Uniform Re-Sampling (Rosenfeld\textsuperscript{1})

Consider a two step reconstruction process:

1. Approximate the Fourier coefficients at equispaced modes
2. Compute a standard (filtered) Fourier partial sum

Basic Premise

$f$ is compactly supported in physical space. Hence, the Shannon sampling theorem is applicable in Fourier space; i.e.,

$$\hat{f}(\omega) = \sum_{k=-\infty}^{\infty} \text{sinc}(\omega - k) \hat{f}_k, \quad \omega \in \mathbb{R}.$$ 

Uniform Re-Sampling – Implementation

We truncate the problem as follows

\[ \hat{f}(\omega_k) \approx \sum_{|\ell| \leq M} \text{sinc}(\omega_k - \ell) \hat{f}_\ell, \quad k = -N, \ldots, N \]

\[
\begin{bmatrix}
\hat{f}(\omega_{-N}) \\
\vdots \\
\hat{f}(\omega_N)
\end{bmatrix}
\approx
\begin{bmatrix}
\text{sinc}(\omega_{-N} + M) & \ldots & \text{sinc}(\omega_{-N} - M) \\
\vdots & \ddots & \vdots \\
\text{sinc}(\omega_N + M) & \ldots & \text{sinc}(\omega_N - M)
\end{bmatrix}
\begin{bmatrix}
\hat{f}(\omega_{-M}) \\
\vdots \\
\hat{f}(\omega_M)
\end{bmatrix}
\]

measurements \( \hat{f} \)  
Sampling system matrix \( A \in \mathbb{R}^{2N+1 \times 2M+1} \)  
re-sampled coefficients \( \bar{f} \)
Uniform Re-Sampling – Implementation

The (equispaced) re-sampled coefficients are approximated as

$$\tilde{f} = A^\dagger \hat{f},$$

where $A^\dagger$ is the Moore-Penrose pseudo-inverse of $A$.

- $A$ and its properties characterize the resulting approximation.

- Regularization may be used (truncated SVD, Tikhonov regularization) in the presence of noise.

- $A^\dagger$ is a dense matrix in general. A block variant of this method exists (Block Uniform Re-Sampling, which constructs a sparse $A^\dagger$.}
Uniform Re-Sampling – Implementation

Figure: Illustration of Block Uniform Re-Sampling

\[^2\text{http://ee-classes.usc.edu/ee591/projects/fall04/zli.pdf}\]
Uniform Re-sampling – Examples

Reconstruction from Polynomial (quadratic) samples.
Uniform Re-sampling – Examples

Reconstruction from Polynomial (quadratic) samples.

Error – Fourier Modes

Reconstruction
Uniform Re-sampling – Examples

Reconstruction from Polynomial (quadratic) samples.

Reconstruction Error

Reconstruction
Further Reading


From Uniform Re-sampling to Convolutional Gridding

Recall that for uniform re-sampling, we use the relation

$$\hat{f}(\omega) = \sum_k \text{sinc}(\omega - k) \hat{f}_k = \left( \hat{f} \ast \text{sinc} \right)(\omega)$$

Since the Fourier transform pair of the sinc function is the box/rect function (of width $2\pi$ and centered at zero), we have

$$f \cdot \Pi \longleftrightarrow \hat{f} \ast \text{sinc}$$

Now consider replacing the sinc function by a bandlimited function $\hat{\phi}$ such that $\hat{\phi}(|\omega|) = 0$ for $|\omega| > q$ (typically a few modes wide). We now have

$$f \cdot \phi \longleftrightarrow \hat{f} \ast \hat{\phi}$$
Convolutional Gridding (Jackson/Meyer/Nishimura . . .)

• Gridding is an inexpensive *direct* approximation to the uniform re-sampling procedure.

• Given measurements \( \hat{f}(\omega_k) \), we compute an approximation to \( \hat{f} \ast \hat{\phi} \) at the equispaced modes using

\[
(\hat{f} \ast \hat{\phi})(\ell) \approx \sum_{|\ell - \omega_k| \leq q} \alpha_k \hat{f}(\omega_k) \hat{\phi}(\ell - \omega_k), \quad \ell = -M, \ldots, M.
\]

• \( \alpha_k \) are desity compensation factors (DCF)s and determine the accuracy of the reconstruction.
Convolutional Gridding (Jackson/Meyer/Nishimura . . . )

Figure: Gridding to a Cartesian Grid

\[^3\text{http://web.eecs.umich.edu/~fessler/papers/files/talk/06/isbi,p2,slide,bw.pdf}\]
Now that we are on equispaced modes, use a (F)DFT to reconstruct an approximation to $f \cdot \phi$ in physical space.

- Recover $f$ by dividing out $\phi$.

- This is typically implemented using a non-uniform FFT.
Why Do We Need Density Compensation?

\[ D_N(x) = \sum_{|k| \leq N} e^{ikx} \]

\[ A_N(x) = \sum_{|k| \leq N} e^{i\omega_k x} \]
Why Do We Need Density Compensation?

Uniform Samples

Quadratic Samples
Density Compensation – Examples

Figure: Voronoi Cells for Radial and Spiral Sampling

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Density Compensation – Examples

Choose $\alpha = \{\alpha_k\}_{-N}^N$ such that

$$\sum_{|k| \leq N} \alpha_k e^{i\omega_k x} \approx \begin{cases} 1 & x = 0 \\ 0 & \text{else} \end{cases}$$

Discretizing on an equispaced grid, we obtain the linear system of equations

$$D\alpha = b,$$

where

- $D_{\ell,j} = e^{i\omega_\ell x_j}$ denotes the (non-harmonic) DFT matrix, and
- $b$ denotes the desired point spread function (Dirac delta).

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Convolutional Gridding – Representative Reconstructions

Reconstruction from Polynomial (quadratic) samples.
Convolutional Gridding – Representative Reconstructions

Reconstruction from Polynomial (quadratic) samples.
Convolutional Gridding – Representative Reconstructions

Reconstruction from Spiral samples (Voronoï weights)\(^4\)

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**True Image (Phantom)**

**Reconstruction**

Convolutional Gridding – Representative Reconstructions

Reconstruction from Spiral samples (Voronoï weights)$^4$

Cross Section

Reconstruction

Further Reading


Non-Uniform Fourier Transforms (Greengard and Lee, 2004)

Non-uniform FFTs efficiently evaluate trigonometric sums of the form

\[(\text{Type I}) \quad F(k) = \frac{1}{N} \sum_{j=0}^{N-1} f_j e^{-ikx_j}, \ x_j \in [0, 2\pi), \ k = -\frac{M}{2}, \ldots, \frac{M}{2} - 1.\]

\[(\text{Type II}) \quad f(x_j) = \sum_{k=-\frac{M}{2}}^{\frac{M}{2}-1} F(k) e^{ikx_j}, \ x_j \in [0, 2\pi).\]

at a computational cost of $O(N \log N + M)$. 

The Type I FFT describes the Fourier coefficients of the function

\[ f(x) = \sum_{j=0}^{N-1} f_j \delta(x - x_j) \]

viewed as a periodic function on \([0, 2\pi]\).

Note that \(f\) is not well resolved by a uniform mesh in \(x\).
Instead, let us compute an approximation to $f_\tau$ defined as

$$f_\tau(x) = (f \ast g_\tau)(x) = \int_0^{2\pi} f(y)g_\tau(x - y)dy,$$

where $g_\tau(x)$ is a periodic one-dimensional heat kernel on $[0, 2\pi]$ given by

$$g_\tau(x) = \sum_{l=-\infty}^{\infty} e^{(x - 2l\pi)^2/4\tau}.$$

$f_\tau$ may be approximated on a uniform grid using

$$f_\tau(2\pi m/M_r) = \sum_{j=0}^{N-1} f_j g_\tau(2\pi m/M_r - x_j).$$
Non-Uniform Fourier Transforms (Greengard and Lee, 2004)

$f_\tau$ is a $2\pi$-periodic $C^\infty$ function and can be well-resolved by a uniform mesh in $x$ whose spacing is determined by $\tau$.

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The Fourier coefficients of \( f_\tau \) can be computed with high accuracy using a standard FFT on an oversampled grid. For example,

\[
F_\tau(k) = \frac{1}{2\pi} \int_0^{2\pi} f_\tau(x)e^{-ikx} \, dx \approx \frac{1}{M_r} \sum_{m=0}^{M_r-1} f_\tau(2\pi m/M_r)e^{-ik2\pi m/M_r}.
\]

We may then obtain \( F(k) \) by a deconvolution; i.e.,

\[
F(k) = \sqrt{\pi/\tau} e^{k^2/\tau} F_\tau(k).
\]

Typical parameters: \( M_r = 2M, \tau = 12/M^2 \). Gaussian spreading of each source to the nearest 24 points yields 12 digits of accuracy.
Other Implementations and Further Reading


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Why are Edges Important?

- Edges are important descriptors of underlying features in a function.

- Edges are often necessary to perform operations such as segmentation and feature recognition.

- Edges may also be incorporated in function reconstruction schemes (for example, spectral re-projection methods).
Detecting Edges from Fourier Data

- Edge detection from Fourier data is non-trivial – it requires the estimation of *local* features from *global* data.

- Applying conventional edge detectors (Sobel, Prewitt, Canny ...) is not optimal – they can pick up Gibbs oscillations as edges.
Edge Detection from Non-Uniform Fourier Data

Two approaches (direct methods)

- Edge detection on re-sampled Fourier data

\[ \hat{f}(\omega_k)_{\omega_k \notin \mathbb{Z}} \xrightarrow{(B)URS} \hat{f}(\ell)_{\ell \in \mathbb{Z}} \xrightarrow{\text{Edge Detection}} \text{Edges} \]

- Edge detection using convolutional gridding

\[ \hat{f}(\omega_k)_{\omega_k \notin \mathbb{Z}} \xrightarrow{\text{Gridding}} (\hat{f} * \hat{\phi})(\ell)_{\ell \in \mathbb{Z}} \xrightarrow{\text{Edge Detection}} \text{Edges} \]

\[ \hat{f}(\omega_k)_{\omega_k \notin \mathbb{Z}} \xrightarrow{\text{Gridding}} ([f] * \hat{\phi})(\ell)_{\ell \in \mathbb{Z}} \xrightarrow{\mathcal{F}^{-1}} \text{Edges} \]
Concentration Method (Gelb, Tadmor)

- Define the *jump function* as follows

\[ [f](x) := f(x^+) - f(x^-) \]

\([f]\) identifies the singular support of \(f\).

- Approximate the singular support of \(f\) using the *generalized conjugate partial Fourier sum*

\[ S_N^\sigma[f](x) = i \sum_{k=-N}^{N} \hat{f}(k) \text{sgn}(k) \sigma \left( \frac{|k|}{N} \right) e^{ikx} \]

- \(\sigma_{k,N}(\eta) = \sigma \left( \frac{|k|}{N} \right)\) are known as *concentration factors*. 
Concentration Method (Gelb, Tadmor)

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S^\sigma_N[f](x) = i \sum_{k=-N}^{N} \hat{f}(k) \text{sgn}(k) \sigma \left( \frac{|k|}{N} \right) e^{ikx}
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- \(\sigma_{k,N}(\eta) = \sigma \left( \frac{|k|}{N} \right)\) are known as *concentration factors*. 
Concentration Method (Gelb, Tadmor)

Admissibility conditions for $\sigma$:

1. $\sum_{k=1}^{N} \sigma \left( \frac{k}{N} \right) \sin(kx)$ is odd.

2. $\frac{\sigma_{k,N}(\eta)}{\eta} \in C^2(0, 1)$

3. $\int_{\epsilon}^{1} \frac{\sigma_{k,N}(\eta)}{\eta} \to -\pi, \quad \epsilon = \epsilon(N) > 0$ being small.
Concentration Method (Gelb, Tadmor)

Theorem (Concentration Property, (Tadmor, Zou))

Assume that $f(\cdot) \in BV[-\pi, \pi]$ is a piecewise $C^2$–smooth function and let $\sigma_{k,N}$ be an admissible concentration factor. Then, $S_N^\sigma[f](x)$ satisfies the concentration property

$$S_N^\sigma[f](x) = \begin{cases} 
O\left(\frac{\log N}{N}\right), & d(x) \lesssim \frac{\log N}{N} \\
O\left(\frac{\log N}{(N d(x))^s}\right), & d(x) \gg \frac{1}{N},
\end{cases}$$

where $d(x)$ denotes the distance between $x$ and the nearest jump discontinuity and $s = s_\sigma > 0$ depends on our choice of $\sigma$. 
## Concentration Factors

<table>
<thead>
<tr>
<th>Factor</th>
<th>Expression</th>
</tr>
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</table>
| **Trigonometric** | \( \sigma_T(\eta) = \frac{\pi \sin(\alpha \eta)}{Si(\alpha)} \)  
\( Si(\alpha) = \int_0^\alpha \frac{\sin(x)}{x} \, dx \) |
| **Polynomial** | \( \sigma_P(\eta) = -p \pi \eta^p \)  
*p is the order of the factor* |
| **Exponential** | \( \sigma_E(\eta) = C\eta \exp\left[\frac{1}{\alpha \eta(\eta-1)}\right] \)  
*C - normalizing constant  
*\( \alpha - order\)  
*C = \frac{\pi}{\int_{1/\sqrt{N}}^{1-1/\sqrt{N}} \exp\left[\frac{1}{\alpha \tau(\tau-1)}\right] d\tau} |

Table: Examples of concentration factors

Figure: Envelopes of Factors in \( k\)-space
Some Examples

(a) Trigonometric Factor

(b) Exponential Factor

Figure: Jump Function Approximation, $N = 128$
Objective

Design a statistically optimal edge detector which accepts a noisy concentration sum approximation and returns a list of jump locations and jump values.

\[ S^o_N[f] \xrightarrow{\text{EDGE DETECTOR}} \{\xi_p\}, \{[f](\xi_p)\} \]
Statistical Formulation

- This is a binary detection theoretic problem – is any given point in the domain an edge (hypothesis $H_1$) or not (hypothesis $H_0$)?

- The Neyman–Pearson lemma tells us that the statistically optimal construction in this case is a correlation detector, which computes correlations of $S_N^\sigma[f]$ with a template waveform.

- Uses a small number of measurements in a neighborhood of the given point\(^5\); for example, to see if the grid point $x_0$ is an edge, use

$$Y = \begin{bmatrix} S_N^\sigma[f](x_0 - h) \\ S_N^\sigma[f](x_0) \\ S_N^\sigma[f](x_0 + h) \end{bmatrix}$$

\(^5\)This will identify the closest grid point to an edge.
Resulting edge detector takes the form

\[ \mathcal{H}_1 : M^T C_V^{-1} Y > \gamma \]

- \( C_V \) is the covariance matrix (depends on the noise characteristics and stencil).
- \( \gamma \) is a threshold which controls the probability of correct detection.
Examples – Edge Detection with Noisy Fourier Data

(a) Noisy Fourier Reconstruction

(b) Jump Detection

Figure: Edge Detection with Noisy Data, $N = 50, \rho = 0.02$, 5−point Trigonometric detector
Examples – Edge Detection with Noisy Fourier Data

(a) Noisy Fourier Reconstruction

(b) Jump Detection

Figure: Edge Detection with Noisy Data, $N = 50, \rho = 0.02, 5$-point Trigonometric detector
Two Dimensional Extensions

For images, apply the method to each dimension separately

\[ S_N^\sigma [f](\bar{x}(\bar{y})) = i \sum_{l=-N}^{N} \text{sgn}(l) \sigma \left( \frac{|l|}{N} \right) \sum_{k=-N}^{N} \hat{f}_{k,l} e^{i(kx+l\bar{y})} \]

(overbar represents the dimension held constant.)
Two Dimensional Extensions

For images, apply the method to each dimension separately

\[
S_N^\sigma[f](x(\bar{y})) = i \sum_{l=-N}^{N} \text{sgn}(l) \sigma \left( \frac{|l|}{N} \right) \sum_{k=-N}^{N} \hat{f}_{k,l} e^{i(kx+l\bar{y})}
\]

(overbar represents the dimension held constant.)
DCF Design for Edge Detection

Choose $\alpha = \{\alpha_k\}_{-N}^{N}$ such that

$$\sum_{|k| \leq N} \alpha_k e^{i\omega_k x} \approx \begin{cases} i \sum_{|\ell| \leq M} \text{sgn}(\ell) \sigma(|\ell|/N) e^{i\ell x} & |x| \leq \pi \\ 0 & \text{else} \end{cases}$$

Discretizing on an equispaced grid, we obtain the linear system of equations

$$D\alpha = \tilde{b},$$

where

- $D$ is the (non-harmonic) DFT matrix with $D_{\ell,j} = e^{i\omega \ell x_j}$, and
- $\tilde{b}$ is a vector containing the values of the generalized conjugate Dirichlet kernel on the equispaced grid.
Numerical Results

Jump Approximation and Corresponding Weights

- $\omega_k$, logarithmically spaced
- $N = 256$ measurements
- Iterative weights solved using LSQR
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Spectral Re-projection

- Spectral reprojection schemes were formulated to resolve the Gibbs phenomenon. They involve reconstructing the function using an alternate basis, $\Psi$ (known as a Gibbs complementary basis).
- Reconstruction is performed using the rapidly converging series

$$f(x) \approx \sum_{l=0}^{m} c_l \psi_l(x), \quad \text{where} \quad c_l = \frac{\langle S_N f, \psi_l \rangle_w}{\|\psi_l\|_w^2}$$

- Reconstruction is performed in each smooth interval. Hence, we require jump discontinuity locations
- High frequency modes of $f$ have exponentially small contributions on the low modes in the new basis
Gegenbauer Reconstruction – Representative Result

Figure: Gegenbauer reconstruction

(e) Reconstruction
(f) Reconstruction error

- Filtered Fourier reconstruction uses 256 coefficients
- Gegenbauer reconstruction uses 64 coefficients
- Parameters in Gegenbauer Reconstruction - $m = 2$, $\lambda = 2$
Some Open Problems

1. Design of Density Compensation Factors and Gridding Windows
2. Exploiting piecewise-smooth structure and edges in reconstruction schemes
3. Parallel imaging
4. Dynamical sampling models and reconstruction schemes for motion corrected imaging