Edge Detection from Spectral Data

With Applications to PSF Estimation and Fourier Reconstruction

Aditya Viswanathan

School of Electrical, Computer and Energy Engineering, Arizona State University
aditya.v@asu.edu

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Prof. Anne Gelb is with the School of Mathematical and Statistical Sciences, Arizona State University

Prof. Douglas Cochran is with the School of Electrical, Computer and Energy Engineering, Arizona State University

Prof. Rosemary Renaut is with the School of Mathematical and Statistical Sciences, Arizona State University

Dr. Wolfgang Stefan is with the Department of Computational and Applied Mathematics, Rice University

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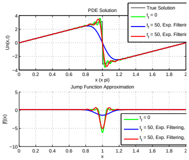
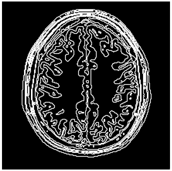
Why do we need edge data?

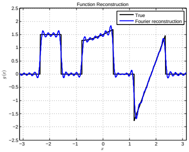
- Solve PDE's with shocks more accurately.
- Identify tissue boundaries in medical images and segment them.
- Reconstruct piecewise-analytic functions from Fourier and other spectral expansion coefficients with uniform and exponential convergence properties.



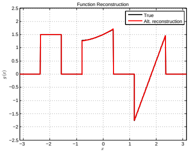
(a) PDE solution with shock discontinuity



(b) MRI tissue boundary detection

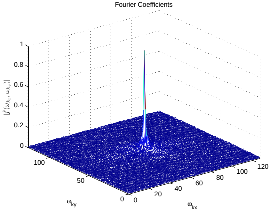


(c) Fourier reconstruction showing Gibbs

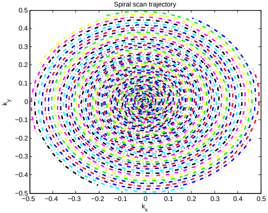


(d) Alt. reconstruction

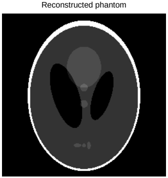
Application – Magnetic Resonance Imaging



(e) Acquired Fourier Samples



(f) Sampling Trajectory



(g) Reconstructed Image

Figure: MR Imaging[a]

[a] Sampling pattern courtesy Dr. Jim Pipe, Barrow Neurological Institute, Phoenix, Arizona

Problem Statement

Objective: To recover location, magnitude and sign of jump discontinuities from a finite number of spectral coefficients

Assumptions:

- f is 2π-periodic and piecewise-smooth function in $[-\pi, \pi)$.
- Its *jump function* is defined as

$$[f](x) := f(x^+) - f(x^-)$$

- It has Fourier series coefficients

$$\hat{f}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} dx\,, \quad k \in [-N, N]$$

A jump discontinuity is a local feature; i.e., the jump function at any point x only depends on the values of f at x^+ and x^-. However, $\hat{f}$ is a global representation; i.e., $\hat{f}(k)$ are obtained using values of f over the entire domain $[-\pi, \pi)$.

Outline

1 Introduction
- Motivation
- Problem Statement

2 Jump detection using the Concentration method
- The Concentration Method
- Concentration Factors
- Statistical Analysis of the Concentration Method
- Iterative Formulations

3 Applications
- PSF Estimation in Blurring Problems
- Applications to Fourier Reconstruction

Getting Jump Data from Fourier Coefficients

Let f contain a single jump at $x = \zeta$.

$$\begin{aligned}
\hat{f}(k) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}dx \\
&= \frac{1}{2\pi}\left(\int_{-\pi}^{\zeta^-} f(x)e^{-ikx}dx + \int_{\zeta^+}^{\pi} f(x)e^{-ikx}dx\right)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2\pi}\left(f(x)\,\frac{e^{-ikx}}{-ik}\bigg|_{-\pi}^{\zeta^-} - \int_{-\pi}^{\zeta^-} f'(x)\frac{e^{-ikx}}{-ik}dx \right. \\
&\qquad \left. + f(x)\,\frac{e^{-ikx}}{-ik}\bigg|_{\zeta+}^{\pi} - \int_{\zeta+}^{\pi} f'(x)\,\frac{e^{-ikx}}{-ik}dx \right)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2\pi}\left(\frac{f(\zeta^-)e^{-ik\zeta} - f(-\pi)e^{ik\pi}}{-ik} - \int_{-\pi}^{\zeta^-} f'(x)\frac{e^{-ikx}}{-ik}dx \right. \\
&\qquad \left. + \frac{f(\pi)e^{-ik\pi} - f(\zeta^+)e^{-ik\zeta}}{-ik} - \int_{\zeta+}^{\pi} f'(x)\frac{e^{-ikx}}{-ik}dx \right)
\end{aligned}$$

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$$\left. + f(x)\,\frac{e^{-ikx}}{-ik}\bigg|_{\zeta^+}^{\pi} - \int_{\zeta^+}^{\pi} f'(x)\,\frac{e^{-ikx}}{-ik}dx\right)$$

$$= \frac{1}{2\pi}\left(\frac{f(\zeta^-)e^{-ik\zeta} - f(-\pi)e^{ik\pi}}{-ik} - \int_{-\pi}^{\zeta^-} f'(x)\frac{e^{-ikx}}{-ik}dx \right.$$
$$\left. + \frac{f(\pi)e^{-ik\pi} - f(\zeta^+)e^{-ik\zeta}}{-ik} - \int_{\zeta^+}^{\pi} f'(x)\frac{e^{-ikx}}{-ik}dx\right)$$

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&\qquad \left. + f(x)\,\frac{e^{-ikx}}{-ik}\bigg|_{\zeta^+}^{\pi} - \int_{\zeta^+}^{\pi} f'(x)\,\frac{e^{-ikx}}{-ik}dx\right) \\
&= \frac{1}{2\pi}\left(\frac{f(\zeta^-)e^{-ik\zeta} - f(-\pi)e^{ik\pi}}{-ik} - \int_{-\pi}^{\zeta^-} f'(x)\frac{e^{-ikx}}{-ik}dx \right. \\
&\qquad \left. + \frac{f(\pi)e^{-ik\pi} - f(\zeta^+)e^{-ik\zeta}}{-ik} - \int_{\zeta^+}^{\pi} f'(x)\frac{e^{-ikx}}{-ik}dx\right)
\end{aligned}$$

Getting Jump Data from Fourier Coefficients

$$\hat{f}(k) = \frac{1}{2\pi}\left(\frac{f(\zeta^-)e^{-ik\zeta} - f(-\pi)e^{ik\pi}}{-ik} - \int_{-\pi}^{\zeta^-} f'(x)\frac{e^{-ikx}}{-ik}dx + \frac{f(\pi)e^{-ik\pi} - f(\zeta^+)e^{-ik\zeta}}{-ik} - \int_{\zeta^+}^{\pi} f'(x)\frac{e^{-ikx}}{-ik}dx\right)$$

$$= \left(f(\zeta^+) - f(\zeta^-)\right)\frac{e^{-ik\zeta}}{2\pi ik} + \frac{f(-\pi)e^{ik\pi} - f(\pi)e^{-ik\pi}}{2\pi ik} + \mathcal{O}\left(\frac{1}{k^2}\right)$$

Since f is periodic, $f(-\pi) = f(\pi)$ and the second term vanishes.

$$\hat{f}(k) = [f](\zeta)\,\frac{e^{-ik\zeta}}{2\pi ik} + \mathcal{O}\left(\frac{1}{k^2}\right)$$

Getting Jump Data from Fourier Coefficients

$$\hat{f}(k) = \frac{1}{2\pi}\left(\frac{f(\zeta^-)e^{-ik\zeta} - f(-\pi)e^{ik\pi}}{-ik} - \int_{-\pi}^{\zeta^-} f'(x)\frac{e^{-ikx}}{-ik}dx + \frac{f(\pi)e^{-ik\pi} - f(\zeta^+)e^{-ik\zeta}}{-ik} - \int_{\zeta^+}^{\pi} f'(x)\frac{e^{-ikx}}{-ik}dx\right)$$

$$= \left(f(\zeta^+) - f(\zeta^-)\right)\frac{e^{-ik\zeta}}{2\pi ik} + \frac{f(-\pi)e^{ik\pi} - f(\pi)e^{-ik\pi}}{2\pi ik} + \mathcal{O}\left(\frac{1}{k^2}\right)$$

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Getting Jump Data from Fourier Coefficients

$$\begin{aligned}\hat{f}(k) &= \frac{1}{2\pi}\left(\frac{f(\zeta^-)e^{-ik\zeta} - f(-\pi)e^{ik\pi}}{-ik} - \int_{-\pi}^{\zeta^-} f'(x)\frac{e^{-ikx}}{-ik}dx\right.\\ &\qquad \left. + \frac{f(\pi)e^{-ik\pi} - f(\zeta^+)e^{-ik\zeta}}{-ik} - \int_{\zeta^+}^{\pi} f'(x)\frac{e^{-ikx}}{-ik}dx\right)\\ &= \left(f(\zeta^+) - f(\zeta^-)\right)\frac{e^{-ik\zeta}}{2\pi ik} + \frac{f(-\pi)e^{ik\pi} - f(\pi)e^{-ik\pi}}{2\pi ik} + \mathcal{O}\left(\frac{1}{k^2}\right)\end{aligned}$$

Since f is periodic, $f(-\pi) = f(\pi)$ and the second term vanishes.

$$\hat{f}(k) = [f](\zeta)\,\frac{e^{-ik\zeta}}{2\pi ik} + \mathcal{O}\left(\frac{1}{k^2}\right)$$

Extracting Jump Information

Let us compute a 'filtered' partial Fourier sum of the form

$$S_N[f](x) = \sum_{k=-N}^{N} \left(\frac{i\pi k}{N}\right) \hat{f}(k) e^{ikx}$$

$$\begin{aligned} S_N[f](x) &= \sum_{k=-N}^{N} \left(\frac{i\pi k}{N}\right) \hat{f}(k) e^{ikx} \\ &= \sum_{k=-N}^{N} \left(\frac{i\pi k}{N}\right) \left[[f](\zeta) \frac{e^{-ik\zeta}}{2\pi ik} + \mathcal{O}\left(\frac{1}{k^2}\right) \right] e^{ikx} \\ &= [f](\zeta) \frac{1}{2N} \sum_{k=-N}^{N} e^{ik(x-\zeta)} + \sum_{k=-N}^{N} \mathcal{O}\left(\frac{1}{k}\right) e^{ik(x-\zeta)} \end{aligned}$$

- First term is a scaled (by the jump value) Dirac delta localized at $x = \zeta$ (the jump location)
- The second term is a manifestation of the global nature of Fourier data

Concentration Method (Gelb, Tadmor)

Generalized conjugate partial Fourier sum:

$$S_N^\sigma[f](x) = i \sum_{k=-N}^{N} \hat{f}(k)\,\mathrm{sgn}(k)\,\sigma\left(\frac{|k|}{N}\right) e^{ikx}$$

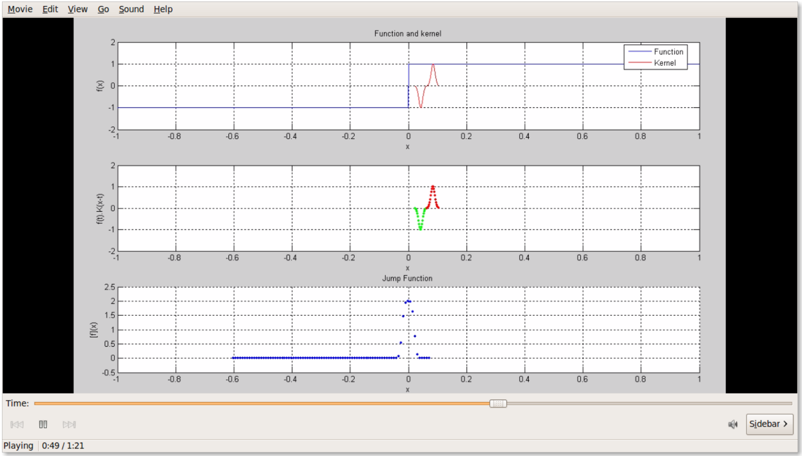
where, $\sigma_{k,N}(\eta) = \sigma(\frac{|k|}{N})$ are known as *concentration factors*. Concentration factors have to satisfy certain properties in order to be admissible. These include,

1. $\sum_{k=1}^{N} \sigma\left(\frac{k}{N}\right) \sin(kx)$ be odd
2. $\frac{\sigma(\eta)}{\eta} \in C^2(0,1)$
3. $\int_\epsilon^1 \frac{\sigma(\eta)}{\eta} \to -\pi$, $\epsilon = \epsilon(N) > 0$ being small

Under these conditions, we have the following relation (concentration property)

$$S_N^\sigma[f](x) = [f](x) + \mathcal{O}(\epsilon), \quad \epsilon = \epsilon(N) > 0 \text{ being small}$$

Illustration of the Concentration Method



Classical Concentration Factors

Factor	Expression
Trigonometric	$\sigma_T(\eta) = \dfrac{\pi \sin(\alpha\,\eta)}{Si(\alpha)}$ $Si(\alpha) = \displaystyle\int_0^{\alpha} \frac{\sin(x)}{x}\,dx$
Polynomial	$\sigma_P(\eta) = -p\,\pi\,\eta^p$ p is the order of the factor
Exponential	$\sigma_E(\eta) = C\,\eta\,\exp\left(\dfrac{1}{\alpha\,\eta\,(\eta-1)}\right)$ C - normalizing constant; α - order $C = \dfrac{\pi}{\int_{\frac{1}{N}}^{1-\frac{1}{N}} \exp\left(\frac{1}{\alpha\,\tau\,(\tau-1)}\right) d\tau}$

Table: Examples of concentration factors

Concentration Factors - Visualization in Fourier Space

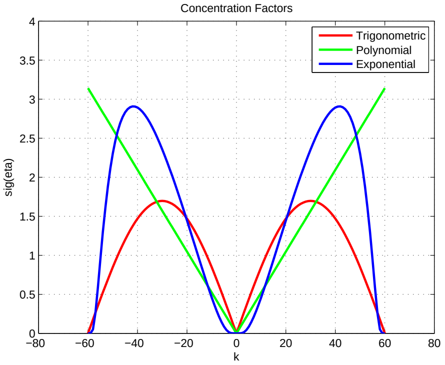
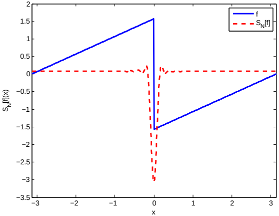
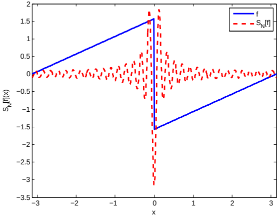


Figure: Envelopes of the Concentration Factors in Fourier Space

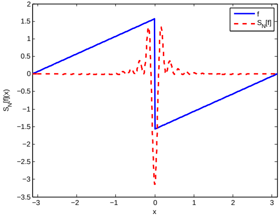
Concentration Factors - Typical Responses



(a) Trigonometric Factor



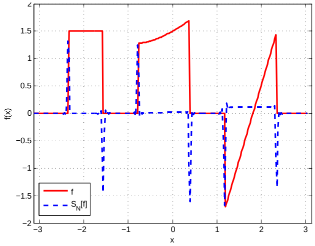
(b) Polynomial Factor



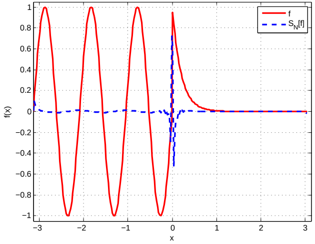
(c) Exponential Factor

Figure: Characteristic Responses of the Concentration Factors to a Sawtooth Function

Examples



(a) Trigonometric Factor



(b) Exponential Factor

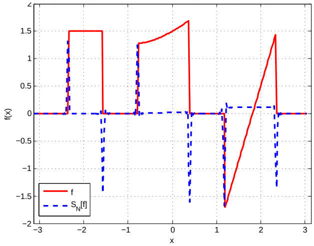
Figure: Jump Response, $N = 128$

- For images, we apply the concentration method to each dimension separately

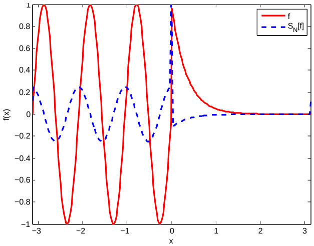
$$S_N^\sigma[f](x(\bar{y})) = i \sum_{l=-N}^{N} \text{sgn}(l)\, \sigma\left(\frac{|l|}{N}\right) \cdot \sum_{k=-N}^{N} \hat{f}_{k,l}\, e^{i(kx+l\bar{y})}$$

where the overbar represents the dimension held constant.

Examples



(a) Trigonometric Factor



(b) Trigonometric Factor

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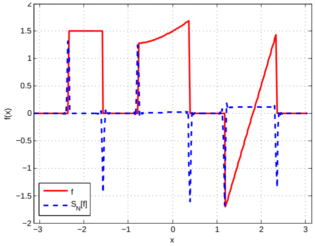

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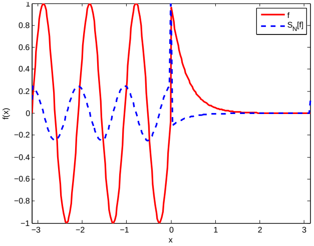

(b) Trigonometric Factor

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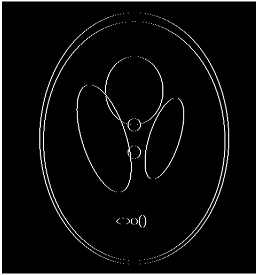


Concentration Factor Design (Viswanathan, Gelb)

- Choose a template function f, eg. the sawtooth function

$$f(x) = \begin{cases} \frac{x-\pi}{2} & x < 0 \\ \frac{x+\pi}{2} & x > 0 \end{cases} \qquad [f](x) = \begin{cases} -\pi & x = 0 \\ 0 & \text{else} \end{cases}$$

- Design σ such that $S_N^\sigma[f](x) = i \sum_{k=-N}^{N} \hat{f}(k) \operatorname{sgn}(k) \, \sigma\left(\frac{|k|}{N}\right) e^{ikx}$ closely matches $[f]$.
- Typical problem formulation

$$\begin{aligned} \min_{\sigma} \quad & \phi_0(\sigma) \\ \text{subject to} \quad & \phi_m(\sigma) = c_m, \\ & \psi_n(\sigma) \leq c_n \end{aligned}$$

where c_m, c_n are constants and the objective and constraints are typically norm measures or the conjugate partial sum evaluated at certain points/intervals in the domain.

Equivalence of Constraints and Admissibility Conditions

1. $\sum_{k=1}^{N} \sigma\left(\frac{k}{N}\right) \sin(kx)$ be odd $\iff$ conjugate kernel real, odd; hence $\sigma(-\eta) = \sigma(\eta)$ with $\sigma(0) = 0$.
2. $\frac{\sigma(\eta)}{\eta} \in C^2(0,1)$ $\iff$ include $\|\sigma\|_2$ or similar smoothness metric in objective.
3. $\int_\epsilon^1 \frac{\sigma(\eta)}{\eta} \to -\pi$, $\epsilon = \epsilon(N) > 0$ $\iff$ $S_N^\sigma[f](\zeta) = [f](\zeta)$, ζ being a point of discontinuity.

(2) is not really an equivalence, but this turns out be practically inconsequential.

Examples

- Problem Formulation 1

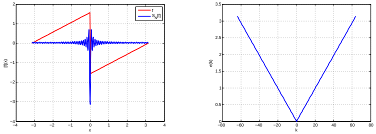
$$\min_{\sigma} \quad \| [f] - S_N^{\sigma}[f] \|_2$$
$$\text{subject to} \quad S_N^{\sigma}[f](0) = -\pi$$

- Problem Formulation 2

$$\min_{\sigma} \quad \| [f] - S_N^{\sigma}[f] \|_1$$
$$\text{subject to} \quad S_N^{\sigma}[f](0) = -\pi$$

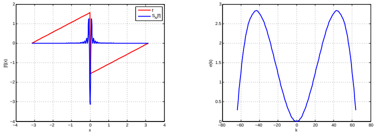
- Problem Formulation 3

$$\min_{\sigma} \quad \| [f] - S_N^{\sigma}[f] \|_1$$
$$\text{subject to} \quad S_N^{\sigma}[f](0) = -\pi$$
$$|S_N^{\sigma}[f](x)| \leq 10^{-3}, \; |x| \in (0.25, 3)$$



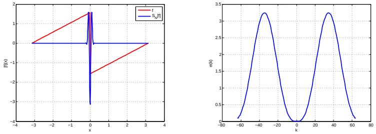
(a) Response (b) Conc. Factor

Figure: Problem Formulation 1, $N = 64$



(a) Response (b) Conc. Factor

Figure: Problem Formulation 2, $N = 64$

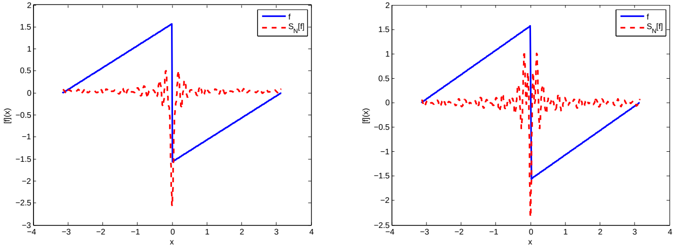


(a) Response (b) Conc. Factor

Figure: Problem Formulation 3, $N = 64$

The Missing Coefficient Problem

- Let us assume that mid-range Fourier coefficients are missing (eg., modes $|30-40|$ for $N=64$).
- Since the response and kernel are real, we assume that both $\hat{f}(\pm p)$ are missing.
- Use of the standard concentration factors results in spurious oscillations in smooth regions.



(a) Response – Trigonometric factor (b) Response – Exponential factor

Figure: Jump approximation with Fourier modes $|30-40|$ missing, $N=64$

Design for Missing Spectral Coefficients

- Choose a template function f, eg. the sawtooth function

$$f(x) = \begin{cases} \frac{x-\pi}{2} & x < 0 \\ \frac{x+\pi}{2} & x > 0 \end{cases} \qquad [f](x) = \begin{cases} -\pi & x = 0 \\ 0 & \text{else} \end{cases}$$

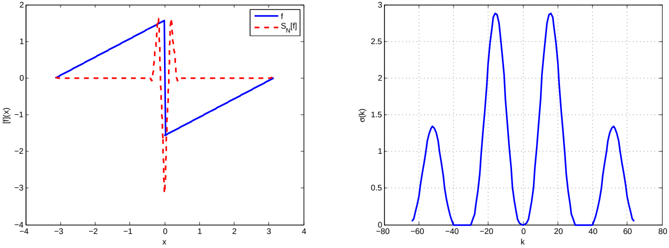
- Explicitly set $\hat{f}(k)\Big|_{30 \le |k| \le 40} = 0$.
- Design σ such that $S_N^\sigma[f](x) = i \sum_{k=-N}^{N} \hat{f}(k)\, \text{sgn}(k)\, \sigma\left(\frac{|k|}{N}\right) e^{ikx}$ closely matches $[f]$.
- Use the standard problem formulation

$$\begin{aligned} \min_{\sigma} \quad & \phi_0(\sigma) \\ \text{subject to} \quad & \phi_m(\sigma) = c_m, \\ & \psi_n(\sigma) \le c_n \end{aligned}$$

where c_m, c_n are constants and the objective and constraints are typically norm measures or the conjugate partial sum evaluated at certain points/intervals in the domain.

An Example

$$\begin{aligned} \min_{\sigma} \quad & \| [f] - S_N^{\sigma}[f] \|_1 \\ \text{subject to} \quad & S_N^{\sigma}[f](0) = -\pi \\ & \sigma(\eta) \geq 0 \\ & |S_N^{\sigma}[f](x)| \leq 10^{-3}, \ |x| \in (0.25, 3) \end{aligned}$$



(a) Response (b) Conc. Factor

Figure: Jump detection with missing spectral data, $N = 64$

Edge detection in noisy data (Viswanathan, Cochran, Gelb, Cates)

- Typically, building an edge detector out of the jump approximation requires comparing the jump function $S_N[f]$ against a threshold
- However, the jump approximations have unique characteristics - mainlobe width, sidelobe oscillations etc.
- Under these circumstances, when deciding whether a point is an edge, it is advantageous to take into consideration measurements in the vicinity of the point
- In the presence of noise, we have to additionally weight the measurements by a covariance matrix
- The final form of the detector (assuming additive white Gaussian noise in Fourier space) is a weighted inner product of the form

$$M^T C_{\mathbf{V}}^{-1} \mathbf{Y} > \gamma$$

where Y is the vector of noisy jump function measurements, M is a template or jump response (noiseless) to a single step edge, $C_{\mathbf{V}}$ is the covariance matrix and γ is a threshold.

The Details

- Assume zero-mean, additive complex white Gaussian noise

$$\hat{\mathbf{g}}(k) = \hat{f}(k) + \hat{\mathbf{v}}(k) \quad \hat{\mathbf{v}}(k) \sim \mathcal{N}[0, \rho^2]$$

- Concentration method is linear, i.e., $S_N^\sigma[g](x) = S_N^\sigma[f](x) + S_N^\sigma[\mathbf{v}](x)$
- Mean: $\mathsf{E}\left[\, S_N^\sigma[\mathbf{g}](x) \,\right] = S_N^\sigma[f](x)$
- Covariance: $(C_{\mathbf{v}})_{p,q}^{x_a, x_b} = \rho^2 \sum_l \sigma_p(\frac{|l|}{N})\sigma_q(\frac{|l|}{N})e^{il(x_a - x_b)}$
- The detection problem is

$$\begin{array}{lcccl} \mathcal{H}_0 : & \mathbf{Y} & = & \mathbf{V} & \sim \mathcal{N}[0, C_{\mathbf{V}}] \\ \mathcal{H}_1 : & \mathbf{Y} & = & M + \mathbf{V} & \sim \mathcal{N}[M, C_{\mathbf{V}}] \end{array}$$

- Solve using Neyman-Pearson Lemma

$$\rightarrow \mathcal{H}_1 : \quad \frac{Pr(\mathbf{Y}|\mathcal{H}_1)}{Pr(\mathbf{Y}|\mathcal{H}_0)} > \gamma$$

- Detector is a generalized matched filter $\rightarrow \mathcal{H}_1 : \quad M^T C_{\mathbf{V}}^{-1} \mathbf{Y} > \gamma$
- $M^T C_{\mathbf{V}}^{-1} M$ is the "SNR" metric and governs detector performance

A Representative Result

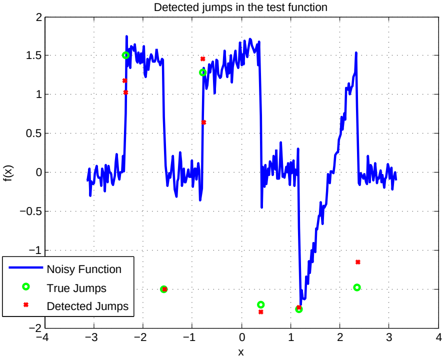


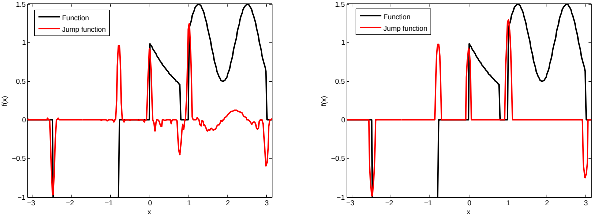
Figure: Results - Edge Detection with Noisy Data, $N = 128, \rho^2 = 7.5, 3-$point Trigonometric detector

- performs well with the exception of false alarms in the vicinity of an edge
- false alarms can be addressed using statistical sidelobe mitigation methods
- Use of multiple concentration factors is possible and indeed encouraged; we use those combinations of concentration factors which yield high SNR

In Search of Precise Jump Locations (Stefan, Viswanathan, Gelb, Renaut)

- Jump approximations using the concentration method are computed using a Fourier partial sum.
- Since the jump function is piecewise-analytic, this approximation suffers from convergence issues.
- In particular, the jump has a minimum, non-zero resolution and there are spurious responses in smooth regions.

Non Linear Post-processing



(a) The minmod response (b) Enhancement of scales

Figure: Non-linear post-processing of the jump approximation

Problem Formulation

In setting up our iterative formulations,

- We will take inspiration from sparsity enforcing regularization routines and their iterative solutions (Tadmor and Zou).
- We will exploit the characteristic responses of the concentration factors in our problem formulation.

We may write

$$S_N^\sigma[f](x) = (f * K_N^\sigma)(x) \approx ([f] * W_N^\sigma)(x)$$

- $S_N^\sigma[f]$ is the jump approximation computed using the concentration method and concentration factor $\sigma(\eta)$.
- W_N^σ is the characteristic response to a unit jump using the concentration factor $\sigma(\eta)$.

Iterative Problem Formulation

$$\min_p \quad \| Wp - S_N[f] \|_2^2 + \lambda \| p \|_1$$

where W is a Toeplitz matrix containing shifted replicates of the characteristic response.

Representative Examples

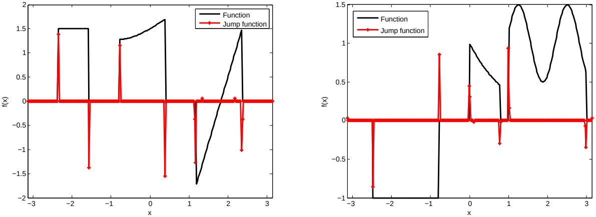


Figure: Jump detection – Iterative Formulation ($N = 40$, Exponential factor)

- Works well with even a small number of measurements
- Works with missing coefficients and non-harmonic Fourier coefficients, i.e., $\hat{f}(\omega_k) := \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(x)e^{-i\omega_k x}dx$
- Can be extended to blurred data ($g = f * h + n$) by modifying the problem formulation

$$\min_p \quad \| H \cdot W \cdot p - S_N[g] \|_2^2 + \lambda \| p \|_1$$

where H is a Toeplitz matrix containing shifted replicates of the blur or psf.

Representative Examples

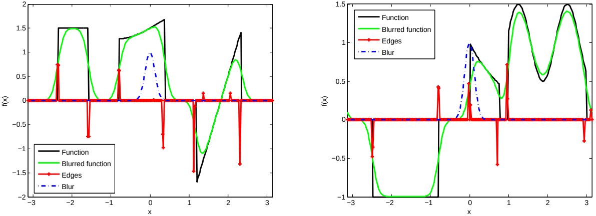


Figure: Jump detection – Iterative Formulation ($N = 128$, Exponential factor, Gaussian Blur)

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Outline

Convolutional Blurring (Viswanathan, Stefan, Cochran, Gelb, Renaut)

- Often, the input data we observe has been subjected to a distorting physical process during measurement, transmission or instrumentation.
- A large class of these distortions can be explained using a convolutional blurring model.
- Corrective actions often require an accurate estimate of the distortion or blur.

The convolutional blurring model can be written as

$$g = f * h + n$$

- f is the true function
- h is the blur or the point-spread function (psf)
- n is noise
- g is the observed function

Let $f \in L^2(-\pi, \pi)$ be piecewise-smooth. We estimate the psf starting with $2N + 1$ blurred Fourier coefficients $\hat{g}(k), k = -N, ..., N$.

PSF Estimation using Edge Detection

Given the blurring model

$$g = f * h + n$$

Principle:

Apply a linear edge detector. We shall assume that the edge detector can be written as a convolution with an appropriate kernel

$$\begin{aligned} S_N^\sigma[g] &= T\,(f * h + n) \\ &= (f * h + n) * K \\ &= f * h * K + n * K \\ &= (f * K) * h + n * K \\ &\approx [f] * h + \tilde{n} \end{aligned}$$

Hence, we observe shifted and scaled replicates of the psf.

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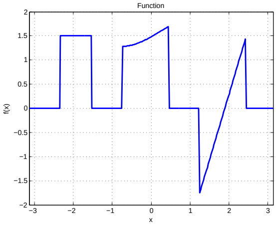
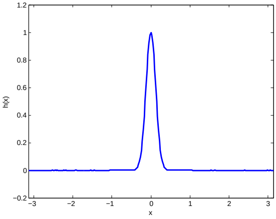
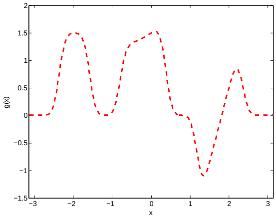
Example (No Noise)



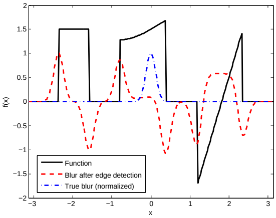
(a) True function



(b) Gaussian Blur PSF



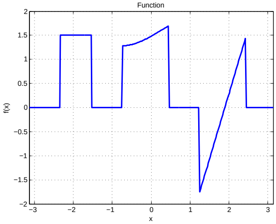
(c) Blurred Function



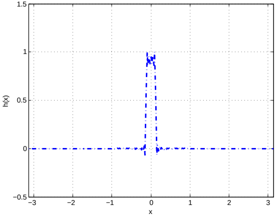
(d) After applying edge detection

Figure: Function subjected to Gaussian blur, $N = 128$

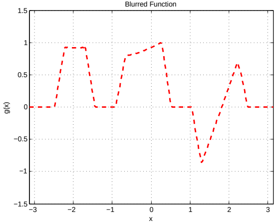
Example (No Noise)



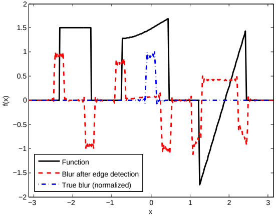
(a) True function



(b) Motion Blur PSF



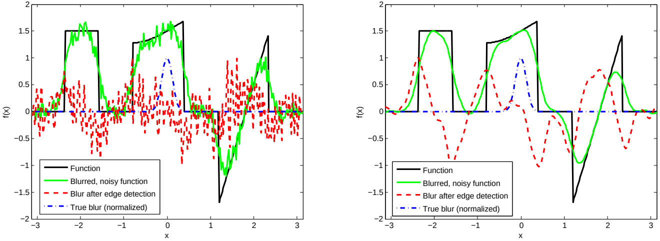
(c) Blurred Function



(d) After applying edge detection

Figure: Function subjected to motion blur, $N = 128$

Representative Examples

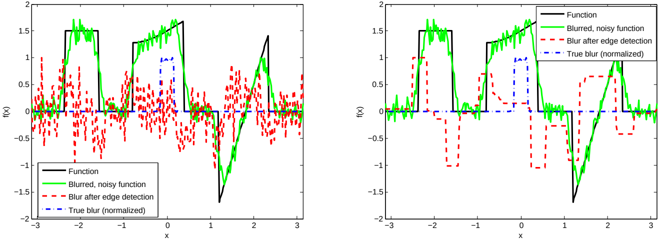


(a) Noisy blur estimation (b) After low-pass filtering

Figure: Function subjected to Gaussian blur, $N = 128$

- Noise distribution – $\hat{n} \sim \mathcal{CN}(0, \frac{1.5}{(2N+1)^2})$
- Second picture subjected to low-pass (Gaussian) filtering
- It is conceivable that the parameter estimation can take into account the effect of Gaussian filtering

Representative Examples



(a) Noisy blur estimation (b) After TV denoising

Figure: Function subjected to Motion blur, $N = 128$

- Cannot perform conventional low-pass filtering since blur is piecewise-smooth
- We compute the noisy blur estimate $S_N[g] \approx [f] * h + n * K_N$
- Denoising problem formulation

$$\min_{p} \quad \| p - S_N[g] \|_2^2 + \lambda \| Lp \|_1$$

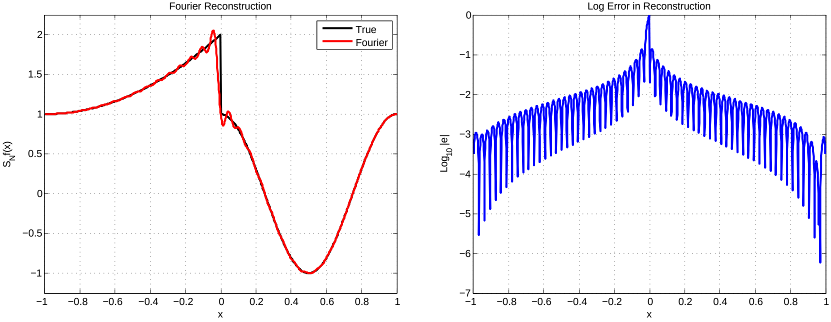
where L is a derivative matrix.

Fourier Reconstruction of Piecewise-smooth functions

$$S_N f(x) = \sum_{k=-N}^{N} \hat{f}(k) e^{ikx}, \quad \hat{f}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} dx$$

Consequences of Gibbs

- Non-uniform convergence – presence of non-physical oscillations in the vicinity of discontinuities
- Reduced order of convergence – first order convergence even in smooth regions of the reconstruction

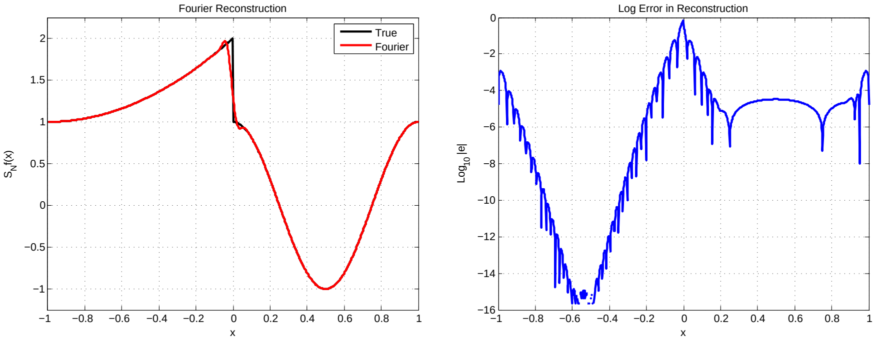


(a) Reconstruction (b) Reconstruction error

Figure: Gibbs Phenomenon, $N = 32$

Filtered Fourier Reconstructions

Filtering helps to ameliorate the effects of Gibbs, but does not eliminate it. In fact, it introduces a smearing artifact in the vicinity of a discontinuity.



(a) Filtered Reconstruction

(b) Reconstruction error

Figure: Exponentially Filtered Reconstruction, $p = 2, N = 64$

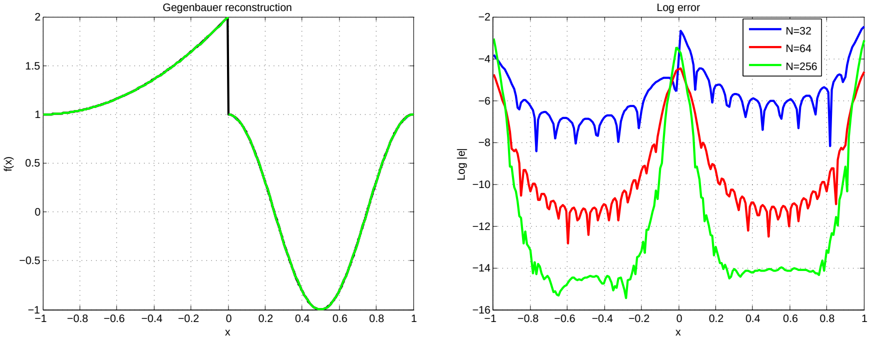
Spectral Re-projection (Gottlieb and Shu)

- Spectral re-projection schemes were formulated to resolve the Gibbs phenomenon. They involve reconstructing the function using an alternate basis, Ψ (known as a Gibbs complementary basis).
- Reconstruction is performed using the rapidly converging series

$$f(x) \approx \sum_{l=0}^{m} c_l \psi_l(x), \quad \text{where} \quad c_l = \frac{\langle f_N, \psi_l \rangle_w}{\|\psi_l\|_w^2},\ f_N \text{ is the Fourier expansion of } f$$

- Reconstruction is performed in each smooth interval. Hence, we require jump discontinuity locations
- High frequency modes of f have exponentially small contributions on the low modes in the new basis

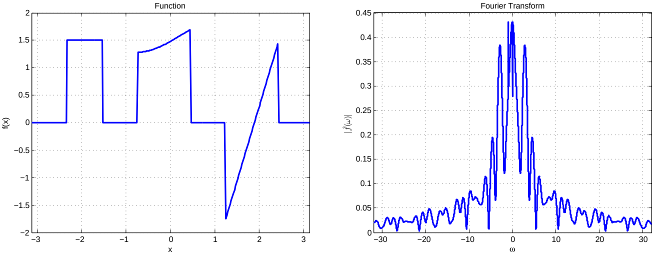
Spectral Re-projection – An Example



(a) Reconstruction (b) Reconstruction error

Figure: Spectral Reprojection Reconstructions

Non-harmonic Fourier Reconstruction (Viswanathan, Gelb, Cochran, Renaut)

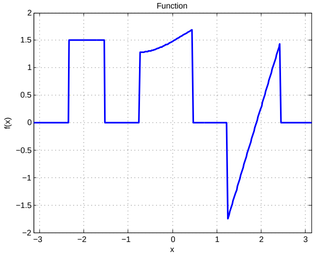


(a) Template Function (b) Fourier Transform

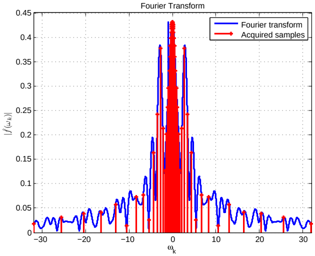
Figure: A motivating example

- Fourier samples violate the quadrature rule for discrete Fourier expansion
- Computational issue – no FFT available
- Mathematical issue – given these coefficients, can we/how do we reconstruct the function?
- Resolution – what accuracy can we achieve given a finite (usually small) number of coefficients?

Non-harmonic Fourier Reconstruction (Viswanathan, Gelb, Cochran, Renaut)



(a) Template Function



(b) Fourier Coefficients, $N = 32$

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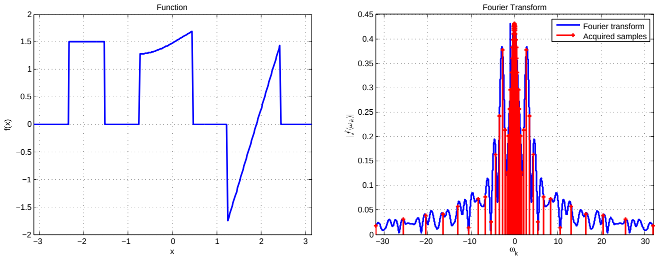

(a) Template Function (b) Fourier Coefficients, $N = 32$

Figure: A template example

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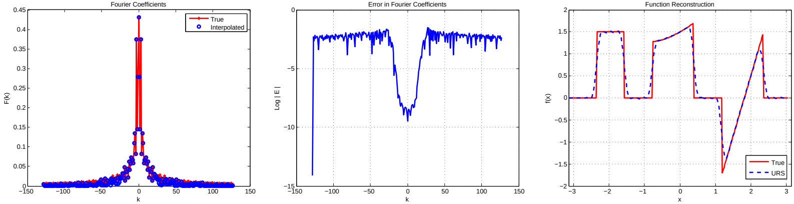
Reconstruction Procedure

1. recover equispaced coefficients $\hat{f}(k)$
2. partial Fourier reconstruction using the FFT algorithm

Equispaced coefficients are obtained by inverting a model derived from application of the sampling theorem.



(a) Recovered Fourier coefficients (b) Error in recovered coefficients (c) Filtered reconstruction

Figure: Reconstruction result, $N = 128$

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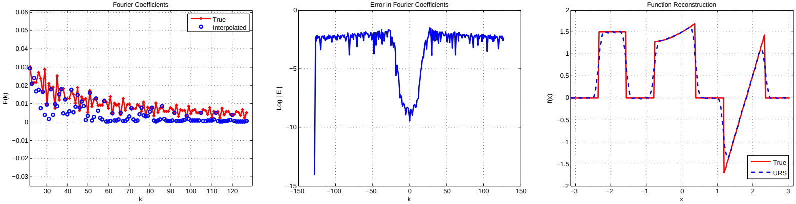

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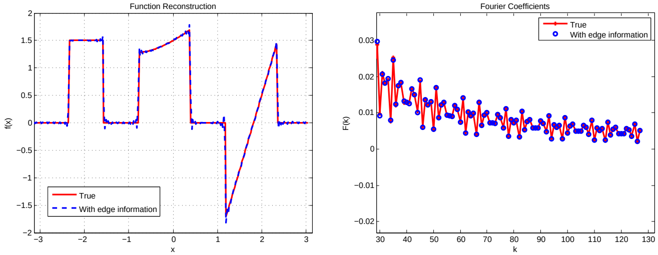
Figure: Reconstruction result, $N = 128$

Incorporating Edge Information

- Compute the high frequency modes using the relation

$$\hat{f}(k) = \sum_{p \in \mathcal{P}} [f](\zeta_p) \, \frac{e^{-ik\zeta_p}}{2\pi i k}$$



(a) Reconstruction - Using edge information

(b) The high modes - Using edge information

Figure: Reconstruction of a test function using edge information

Summary

- We summarized the concentration method of edge detection.
- We looked at routines to design concentration factors.
- We showed iterative routines for accurate detection of edges.
- We briefly surveyed the statistical properties of concentration edge detection.
- We looked at applications of edge detection to
  - Point spread function estimation in blurring problems.
  - Fourier reconstruction of piecewise-smooth functions – spectral re-projection.
  - Non-harmonic Fourier reconstruction.

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Concentration Method

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Spectral Re-projection

1. D. Gottlieb and C.W. Shu, *On the Gibbs phenomenon and its resolution*, in SIAM Review (1997), pp. 644–668.