Edge Detection from Spectral Data with Application to PSF Estimation

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Introduction

Detection of jump discontinuities constitutes an important task in several areas of engineering and signal processing. In certain applications such as MR imaging, input data is collected in the spectral domain. In such cases, the *concentration method* allows us to compute jump locations and values from a finite number of spectral coefficients. In this poster,

- We discuss the design of concentration factors, which determine the characteristics of the jump approximation.
- We provide *waveform-aware* iterative formulations of the method for accurate jump detection.
- We discuss an application of the method to estimating point-spread functions (psf) in blurring problems.

The Concentration Method

Let f be a 2π-periodic piecewise-smooth function. We define its *jump function* as

$$[f](x) := f(x^+) - f(x^-)$$

Note that the jump function is non-zero only at the singular support of f. Given the Fourier coefficients $\{\hat{f}(k)\}_{k=-N}^{N}$, the concentration method, [1,2], computes an approximation to the jump function using the following partial sum

$$S_N^\sigma[f](x) = i \sum_{|k|\leq N} \hat{f}(k)\,\mathrm{sgn}(k)\,\sigma\left(\frac{|k|}{N}\right) e^{ikx}$$

where $\sigma(\eta) = \sigma\left(\frac{|k|}{N}\right)$ are known as concentration factors. There factors are known to satisfy certain admissibility conditions:

1. $\sum_{k=1}^{N} \sigma\left(\frac{k}{N}\right) \sin(kx)$ is odd
2. $\frac{\sigma(\eta)}{\eta} \in C^2(0,1)$
3. $\int_\epsilon^1 \frac{\sigma(\eta)}{\eta} \to -\pi$, $\epsilon = \epsilon(N) > 0$ being small

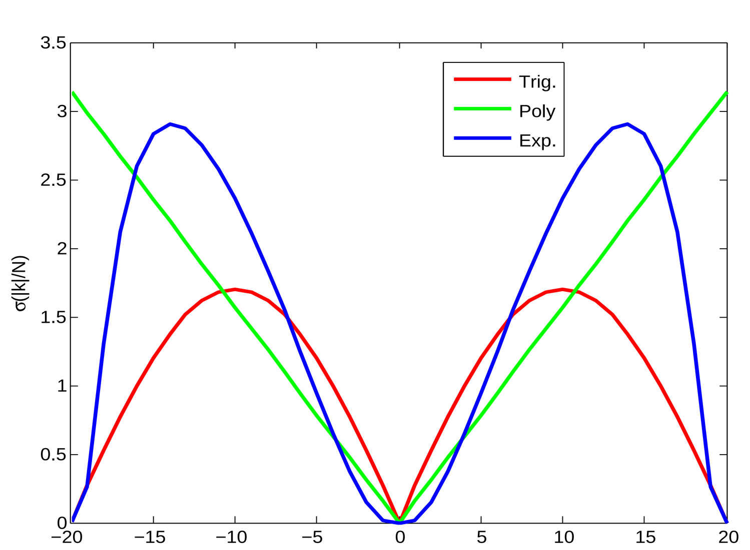

Fig.: Sample Factors

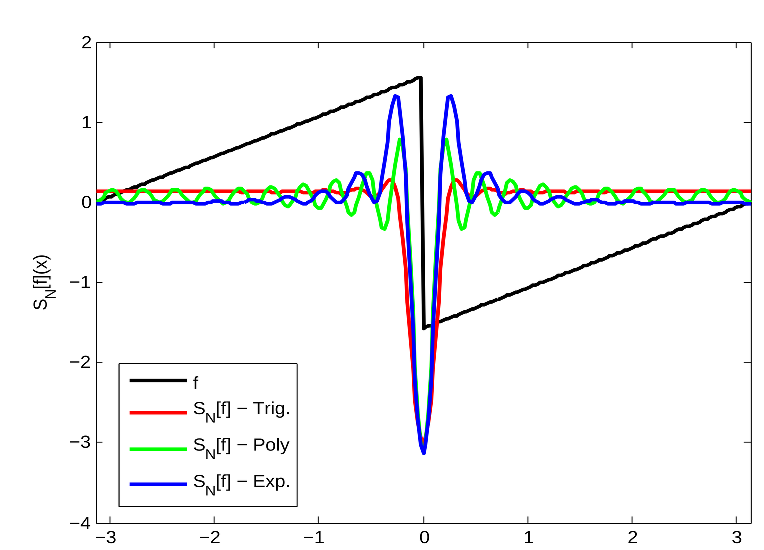

Fig.: Jump Responses, $N = 20$

Concentration Factor Design

Integrating the Fourier integral by parts, we obtain

$$\hat{f}(k) = \frac{1}{2\pi} \sum_{a\in\mathcal{A}} \left(\frac{[f](\zeta_a)}{ik} + \frac{[f'](\zeta_a)}{(ik)^2} + \frac{[f''](\zeta_a)}{(ik)^3} + \ldots \right) e^{-ik\zeta_a}$$

where ζ_a, $a \in \mathcal{A}$ denote the jump locations. Substituting this expression in the concentration sum, we obtain

$$S_N^\sigma[f](x) = ([f] * D_0^{\sigma,N})(x) + ([f'] * D_1^{\sigma,N})(x) + ([f''] * D_2^{\sigma,N})(x) + \ldots$$

with $D_v^{\sigma,N}(\eta) = \frac{1}{2\pi} \sum_{0<|k|\leq N} \frac{i\,\sigma\left(\frac{|k|}{N}\right)\mathrm{sgn}(k)}{(ik)^{v+1}} e^{ik\eta}, \quad v = 0, 1, 2, \ldots$

- Design Objective: enhance $D_0^{\sigma,N}(x)$; reduce $D_p^{\sigma,N}(x)$, $p > 0$.

Sample Problem Formulations

P1. $\min_\sigma \; \| D_0^{\sigma,N} - \delta \|_2$ subject to $D_0^{\sigma,N}\big|_{x=0} = 1$

P2. $\min_\sigma \; \| D_0^{\sigma,N} - \delta \|_1$ subject to $D_0^{\sigma,N}\big|_{x=0} = 1$, $\left|D_1^{\sigma,N}\right|_\infty \leq 10^{-1}$, $\left|D_2^{\sigma,N}\right|_\infty \leq 10^{-3}$, $\left|D_3^{\sigma,N}\right|_\infty \leq 10^{-4}$, $\sigma \geq 0$

P3. (*missing coefficients*) $\min_\sigma \; \| D_0^{\sigma,N} - \delta \|_1$ subject to $D_0^{\sigma,N}\big|_{x=0} = 1$, $\left|D_0^{\sigma,N}(x)\right|_{|x|\geq .35} \leq 10^{-3}$, $\sigma \geq 0$, $\sigma(1) = 0$, $\sigma(\mathbb{K}) = 0$, $\left|\Im\{D_0^{\sigma,N}\}\right|_\infty \leq 10^{-4}$, for $\mathbb{K} \subset [-N, N]$

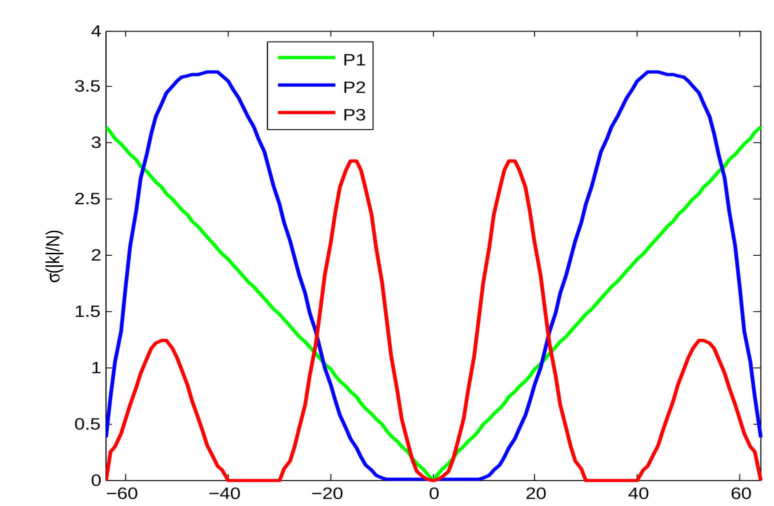

Fig.: Plot of the Factors

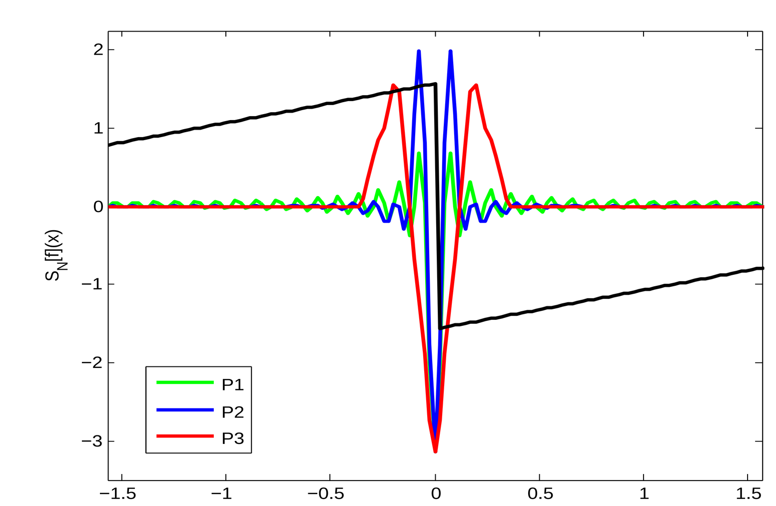

Fig.: Jump Responses, $N = 64$

Iterative "Waveform-aware" Edge Detection

- We take inspiration from sparsity enforcing regularization routines and their iterative solutions, [3].
- We exploit the characteristic responses of the concentration factors.

$$S_N^\sigma[f](x) = (f * K_N^\sigma)(x) \approx ([f] * W_N^\sigma)(x)$$

Here, W_N^σ is the characteristic response to a unit jump.

Iterative Problem Formulation

$$\min_p \; \| W p - S_N[f] \|_2^2 + \lambda \| p \|_1$$

where W is a toeplitz matrix built out of the characteristic response.

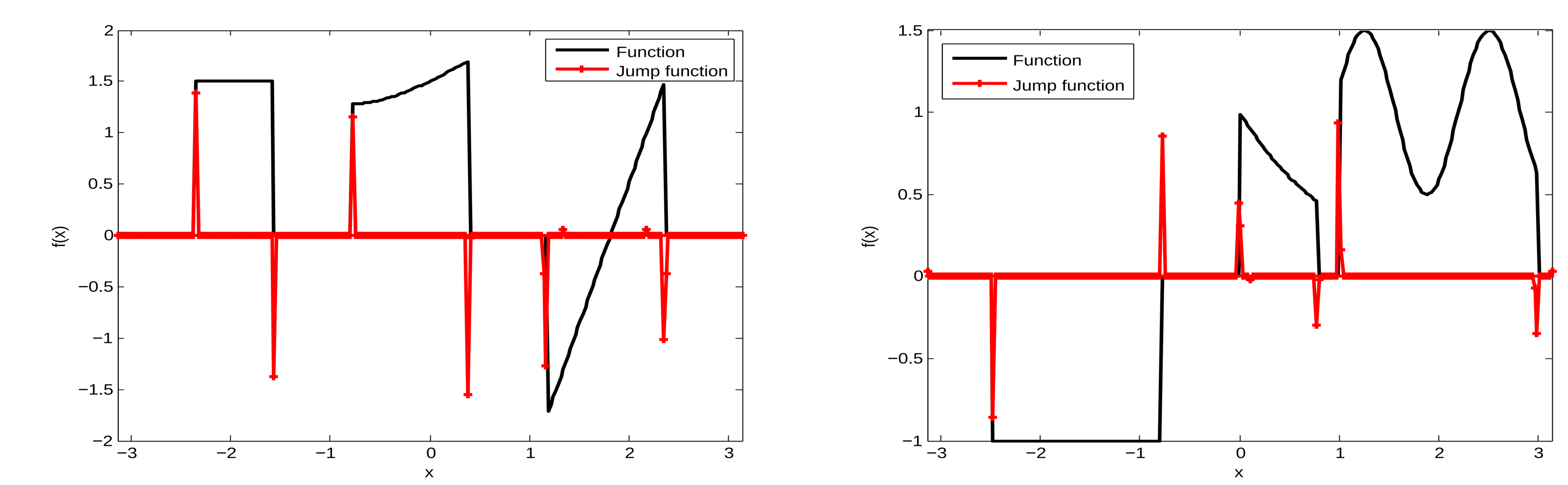

Fig.: Jump detection – Iterative Formulation ($N = 40$, Exponential factor)

PSF Estimation in Blurring Problems

Consider the convolutional blurring model

$$g = f * h + n$$

where f is the true function, h is the unknown blur or the point-spread function (psf), n is noise and g is the observed function. Let us apply the concentration edge detector. We have,

$$\begin{aligned} S_N^\sigma[g] &= T(f * h + n) = (f * h + n) * K_N^\sigma \\ &= f * h * K_N^\sigma + n * K_N^\sigma = (f * K_N^\sigma) * h + n * K_N^\sigma \\ &\approx [f] * h + n * K_N^\sigma \end{aligned}$$

Hence, we observe shifted and scaled replicates of the psf.

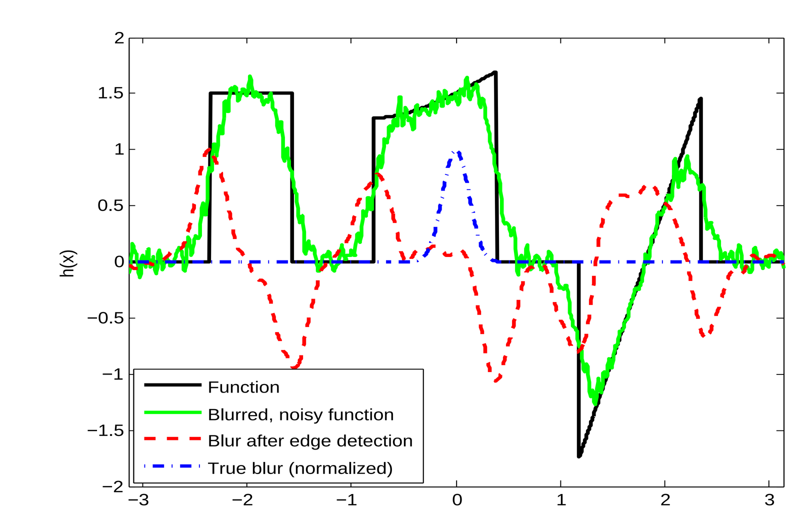

Fig.: Gaussian Blur Estimation

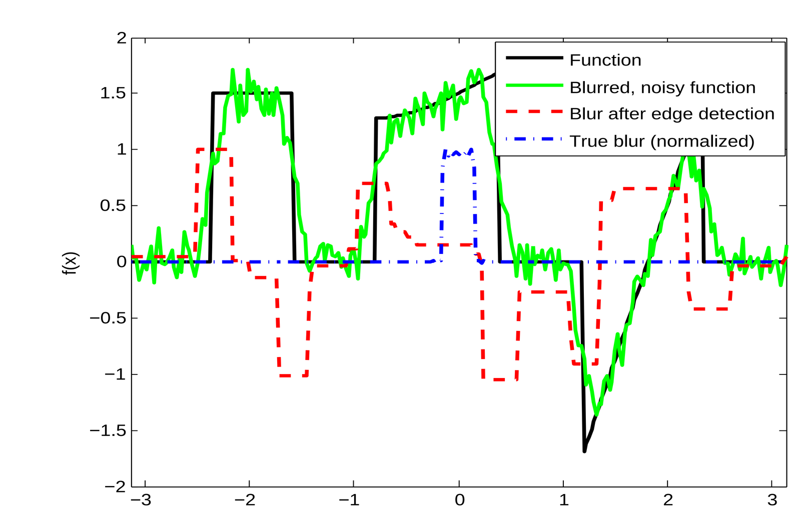

Fig.: Motion Blur Estimation

- Gaussian blur pre-processed using a low-pass filter.
- Motion blur post-processed using TV-denoising.
- Blur parameters may be estimated using simple data fit routines.

References and Acknowledgement

1. A. Gelb and E. Tadmor, *Detection of Edges in Spectral Data*, in Appl. Comp. Harmonic Anal., 7 (1999), pp. 101–135.
2. A. Gelb and E. Tadmor, *Detection of Edges in Spectral Data II. Nonlinear Enhancement*, in SIAM J. Numer. Anal., Vol. 38, 4 (2000), pp. 1389–1408.
3. E. Tadmor and J. Zou, *Novel edge detection methods for incomplete and noisy spectral data*, in J. Fourier Anal. Appl., Vol. 14, 5 (2008), pp. 744–763.

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