Fast Angular Synchronization for Phase Retrieval via Incomplete Information

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Wavelets and Sparsity XVI
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Joint work with

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The Phase Retrieval Problem

$$\text{find} \quad \mathbf{x} \in \mathbb{C}^d \quad \text{given} \quad |M\mathbf{x}|^2 = \mathbf{b} \in \mathbb{R}^D,$$

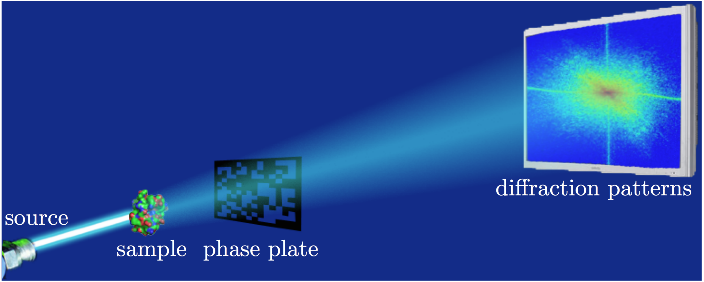
where

- $\mathbf{b} \in \mathbb{R}^D$ are the magnitude or intensity measurements.
- $M \in \mathbb{C}^{D \times d}$ is a measurement matrix associated with these measurements.

Let $\mathcal{A} : \mathbb{R}^D \to \mathbb{C}^d$ denote the recovery method.

The phase retrieval problem involves designing measurement matrix and recovery method pairs, $(M, \mathcal{A})$.

Motivating Applications



From "Phase Retrieval from Coded Diffraction Patterns" by E. J. Candes, X. Li, and M. Soltanolkotabi.

Important applications of Phase Retrieval

- X-ray crystallography
- Diffraction imaging
- Transmission Electron Microscopy (TEM)

In many molecular imaging applications, the detectors only capture intensity measurements.

Existing Computational Approaches

- Alternating projection methods
  [Fienup, 1978], [Fannjiang and Liao, 2012], . . .
- Methods based on semidefinite programming
  PhaseLift [Candes et al., 2012], PhaseCut [Waldspurger et al., 2012], . . .
- Others
  - Frame-theoretic, graph based algorithms [Alexeev et al., 2014]
  - (Stochastic) gradient descent [Candes et al., 2014]
  - [Philipp, 2014], . . .

Today...

- Summarize recently introduced essentially linear-time phase retrieval algorithm based on (deterministic[1]) block-circulant measurement constructions. This algorithm requires solving an *angular synchronization problem*.

  The Angular Synchronization Problem

  Estimate d unknown angles $\theta_1, \theta_2, \ldots, \theta_d \in [0, 2\pi)$ from noisy and possibly incomplete measurements of their differences,

  $$\Delta\theta_{ij} := \theta_i - \theta_j \mod 2\pi.$$

- Discuss fast angular synchronization algorithms when we have structured but highly incomplete phase difference measurements.

[1] for "flat" vectors

Outline

1. The Phase Retrieval Problem
2. BlockPR: Fast Phase Retrieval from Block-Circulant Measurements
3. Angular Synchronization
4. Numerical Simulations

Block-Circulant Measurements

[Iwen, V. and Wang, 2015 (arXiv:1501.02377)]

Block-circulant measurement constructions arise when we acquire correlation (or convolution) type measurements with compactly supported masks.

- Let $\mathbf{x} = [x_1 \ x_2 \ \dots \ x_d]^T \in \mathbb{C}^d$ be the unknown signal.
- Let $(\mathbf{p}_i) = [(\mathbf{p}_i)_1 \ (\mathbf{p}_i)_2 \ \dots \ (\mathbf{p}_i)_\delta \ 0 \ \dots 0]^T$ be the i^{th} mask. Observe that only its first δ entries are non-zero.
- Suppose we are given (squared) correlation measurements

$$\mathbf{b}_i = \left|\texttt{corr}(\mathbf{p}_i, \mathbf{x})\right|^2, \qquad i = 1, 2, \dots, N$$

corresponding to N distinct masks.

Block-Circulant Measurements

[Iwen, V. and Wang, 2015 (`arXiv:1501.02377`)]

Explicitly writing out each measurement, we have

$$(b_i)_\ell = \left| \sum_{k=1}^{\delta} (\mathbf{p_i})_k^* \cdot x_{\ell+k-1} \right|^2, \qquad \ell \in 1, 2, \dots, d, \quad i = 1, 2, \dots, N$$

$$= \sum_{j,k=1}^{\delta} (\mathbf{p_i})_j \, (\mathbf{p_i})_k^* \, x_{\ell+j-1} \, x_{\ell+k-1}^*.$$

This is a system of **linear equations** for the phase differences $x_j x_k^*$.

Example: $d = 4$, $\delta = 2$.

Writing out the correlation sum, we obtain the linear system[2]

$$M'\,\mathbf{x}' = \mathbf{b},$$

where

$$\mathbf{x}' = \left[|x_1|^2 \;\; x_1x_2^* \;\; x_2x_1^* \;\; |x_2|^2 \;\; x_2x_3^* \;\; x_3x_2^* \;\; |x_3|^2 \;\; x_3x_4^* \;\; x_4x_3^* \;\; |x_4|^2 \;\; x_4x_1^* \;\; x_1x_4^*\right]^T,$$

$$\mathbf{b} = \left[(b_1)_1 \;\; (b_2)_1 \;\; (b_3)_1 \;\; (b_1)_2 \;\; (b_2)_2 \;\; (b_3)_2 \;\; (b_1)_3 \;\; (b_2)_3 \;\; (b_3)_3 \;\; (b_1)_4 \;\; (b_2)_4 \;\; (b_3)_4\right]^T,$$ and

$$M' = \begin{bmatrix}
(\mathbf{p}_1)_{1,1} & (\mathbf{p}_1)_{1,2} & (\mathbf{p}_1)_{2,1} & (\mathbf{p}_1)_{2,2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(\mathbf{p}_2)_{1,1} & (\mathbf{p}_2)_{1,2} & (\mathbf{p}_2)_{2,1} & (\mathbf{p}_2)_{2,2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(\mathbf{p}_3)_{1,1} & (\mathbf{p}_3)_{1,2} & (\mathbf{p}_3)_{2,1} & (\mathbf{p}_3)_{2,2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & (\mathbf{p}_1)_{1,1} & (\mathbf{p}_1)_{1,2} & (\mathbf{p}_1)_{2,1} & (\mathbf{p}_1)_{2,2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & (\mathbf{p}_2)_{1,1} & (\mathbf{p}_2)_{1,2} & (\mathbf{p}_2)_{2,1} & (\mathbf{p}_2)_{2,2} & 0 & 0 & 0 & 0 & 0 \\
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\end{bmatrix}.$$

For large d, we can solve for $\mathbf{x}'$ using FFTs!

[2] we have used the notation $(\mathbf{p_i})_{j,k} := (\mathbf{p_i})_j (\mathbf{p_i})_k^*$.

Block-Circulant Matrix: Condition Number Bounds

Theorem (Iwen, V., Wang 2015)

Choose entries of the measurement mask $(\mathbf{p}_i)$ *as follows:*

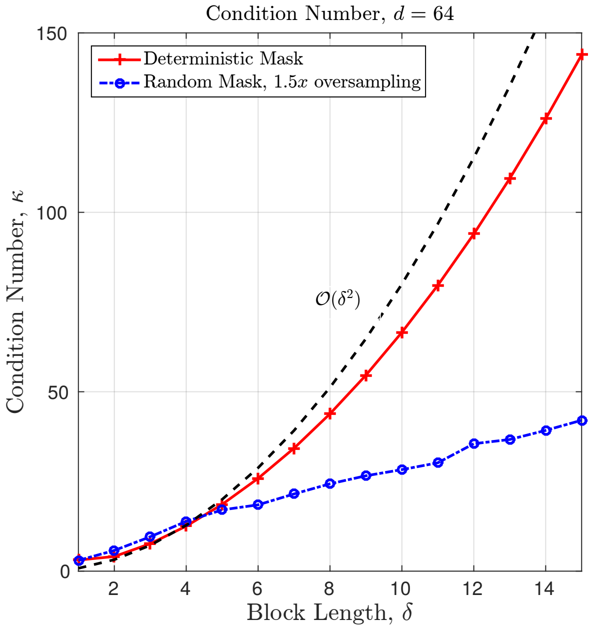
$$(\mathbf{p}_i)_\ell = \begin{cases} \frac{\mathrm{e}^{-\ell/a}}{\sqrt[4]{2\delta-1}} \cdot \mathrm{e}^{\frac{2\pi \mathrm{i} \cdot i \cdot \ell}{2\delta-1}}, & \ell \leq \delta \\ 0, & \ell > \delta \end{cases}, \qquad a := \max\left\{4, \tfrac{\delta-1}{2}\right\}, \quad i = 1, 2, \ldots, N.$$

Then, the resulting system matrix for the phase differences, M'*, has condition number*

$$\kappa(M') < \max\left\{144\mathrm{e}^2, \frac{9\mathrm{e}^2}{4} \cdot (\delta - 1)^2\right\}.$$

- Deterministic (windowed DFT-type) measurement masks!
- δ is typically chosen to be $c \log_2 d$ with c small (2–3).
- Extensions: oversampling, random masks ....

Block-Circulant Matrix: Condition Number Bounds



Outline

Back to our Example ...

$$\left[|x_1|^2\ x_1x_2^*\ x_2x_1^*\ |x_2|^2\ x_2x_3^*\ x_3x_2^*\ |x_3|^2\ x_3x_4^*\ x_4x_3^*\ |x_4|^2\ x_4x_1^*\ x_1x_4^*\right]^T$$

$\downarrow$ (re-arrange)

$$\begin{bmatrix} |x_1|^2 & x_1x_2^* & 0 & x_1x_4^* \\ x_2x_1^* & |x_2|^2 & x_2x_3^* & 0 \\ 0 & x_3x_2^* & |x_3|^2 & x_3x_4^* \\ x_4x_1^* & 0 & x_4x_3^* & |x_4|^2 \end{bmatrix}$$ ($2\delta - 1$ entries in band)

$\downarrow$ (normalize)

$$\begin{bmatrix} 1 & e^{\mathbb{i}(\phi_1-\phi_2)} & 0 & e^{\mathbb{i}(\phi_1-\phi_4)} \\ e^{\mathbb{i}(\phi_2-\phi_1)} & 1 & e^{\mathbb{i}(\phi_2-\phi_3)} & 0 \\ 0 & e^{\mathbb{i}(\phi_3-\phi_2)} & 1 & e^{\mathbb{i}(\phi_3-\phi_4)} \\ e^{\mathbb{i}(\phi_4-\phi_1)} & 0 & e^{\mathbb{i}(\phi_4-\phi_3)} & 1 \end{bmatrix}$$

$\downarrow$ (angular synchronization)

$$\phi_1, \phi_2, \phi_3, \phi_4$$

(Signal Reconstruction) $\left[|x_1|e^{\mathbb{i}\,\phi_1}\quad |x_2|e^{\mathbb{i}\,\phi_2}\quad |x_3|e^{\mathbb{i}\,\phi_3}\quad |x_4|e^{\mathbb{i}\,\phi_4}\right]^T$

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$$\left[|x_1|^2 \; x_1x_2^* \; x_2x_1^* \; |x_2|^2 \; x_2x_3^* \; x_3x_2^* \; |x_3|^2 \; x_3x_4^* \; x_4x_3^* \; |x_4|^2 \; x_4x_1^* \; x_1x_4^*\right]^T$$

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The Angular Synchronization Problem

After solving the block-circulant system, we recover the diagonal entries of the rank one matrix $\mathbf{x}\mathbf{x}^* \in \mathbb{C}^{d \times d}$;

$$\text{i.e.,} \qquad (X')_{j,k} = \begin{cases} (\mathbf{x}\mathbf{x}^*)_{j,k} & \text{if } |j-k \bmod d| < \delta \\ 0 & \text{otherwise.} \end{cases} .$$

Normalizing each entry of X', we obtain the matrix $A \in \mathbb{C}^{d \times d}$ with

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where $x_j = C_j \mathbb{e}^{\mathbb{i}\phi_j}$ for $j = 1, 2, \ldots, d$.

The Angular Synchronization problem: Given $A \in \mathbb{C}^{d \times d}$, find (upto a global phase offset) $\phi_1, \ldots, \phi_d \in [0, 2\pi)$, or equivalently, the vector $\tilde{\mathbf{x}} \in \mathbb{C}^d$ with

$$(\tilde{x})_j := \mathbb{e}^{\mathbb{i}\phi_j} \;=\; x_j/|x_j|.$$

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Previous Work

- Eigenvector and Semidefinite Programming based methods (Singer 2011, Bandeira et al. 2013, Bandeira et al. 2014) Analysis under the assumption that phase difference measurements are missing at random.
- Others: Greedy methods (Iwen et al. 2015), Alternating projection algorithms . . .

What's different with our realization of the problem?

$$(A)_{j,k} = \begin{cases} e^{i(\phi_j - \phi_k)} & \text{if } |j - k \mod d| < \delta \\ 0 & \text{otherwise,} \end{cases},$$

Structured, highly incomplete phase difference data

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Structured, highly incomplete phase difference data

Spectral Properties of A

Lemma

The largest eigenvalue of A is $\lambda_1 = 2\delta - 1$, and $\tilde{\mathbf{x}}$ is an eigenvector of A with this maximal eigenvalue.

The leading eigenvector is indeed the desired solution of this angular synchronization problem.

Note:

$$(\tilde{x})_j := \mathbb{e}^{\mathbb{i}\phi_j} \ = \ x_j/|x_j|.$$

Spectral Properties of A

Lemma

The smallest eigenvalue of A has magnitude $|\lambda_d| \leq 2\delta - 3 = \lambda_1 - 2$ for all $\delta \geq 3$.

Lemma

The largest two eigenvalues of A are not equal, $\lambda_1 \neq \lambda_2$, for all $\delta \geq 2$.

- The top eigenvector solution is unique.
- $\lambda_1 > |\lambda_d|$.

Spectral Properties of A

Lemma

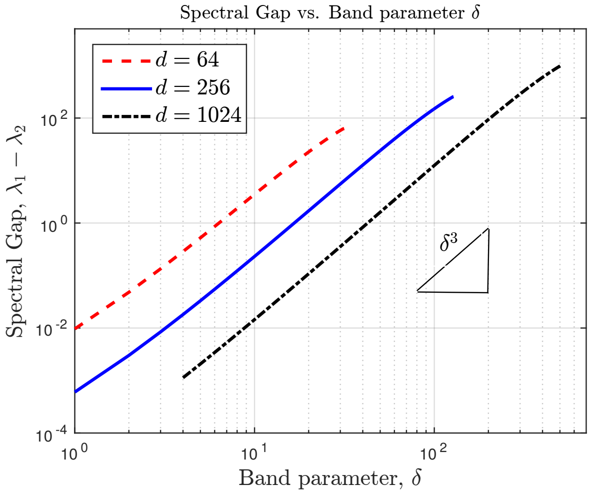
Let $c \in \mathbb{R}^+$. We will have $c\sqrt{\sum_{j=2}^{d} \lambda_j^2} \leq \lambda_1$ *whenever* $\delta \geq \frac{1+(d+1)c^2}{2(1+c^2)}$.

The largest eigenvalue will contain a substantial fraction of the spectral energy of A if δ is taken to be sufficiently large.

Approximate $\widetilde{\mathbf{x}}$ by the leading eigenvector of A. This can be computed efficiently using the power method.

Outline

Spectral Properties of A: Spectral Gap vs δ



- $\widetilde{\mathbf{x}}$ is iid standard Complex Gaussian.
- Averaged over 100 trials.
- Computed using Matlab's `eigs`.

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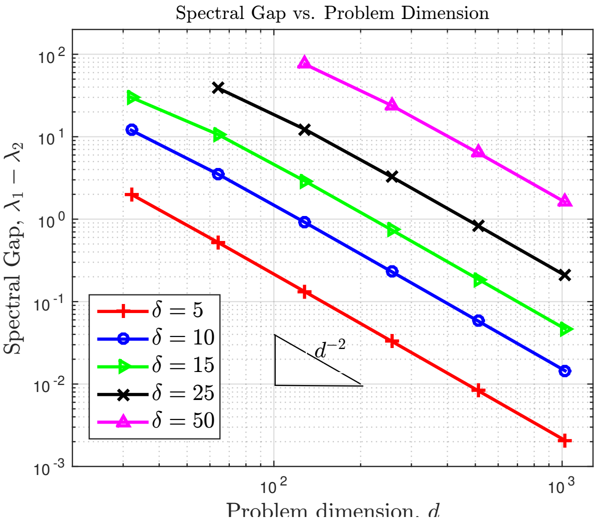

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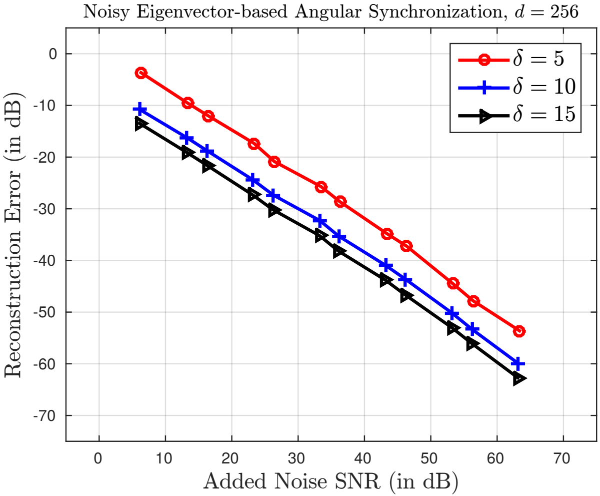
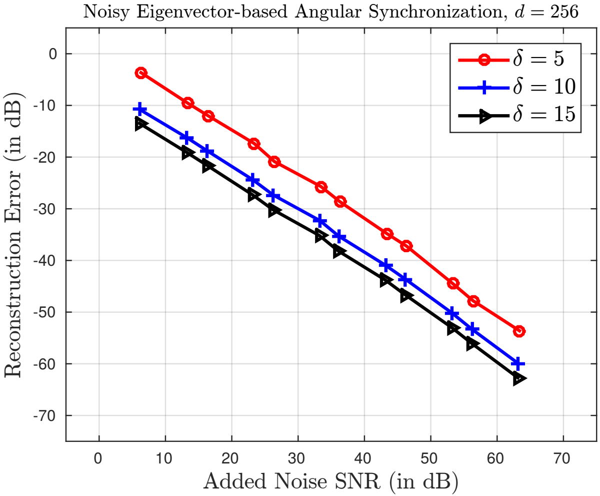
Noisy Angular Synchronization



Noise Model:

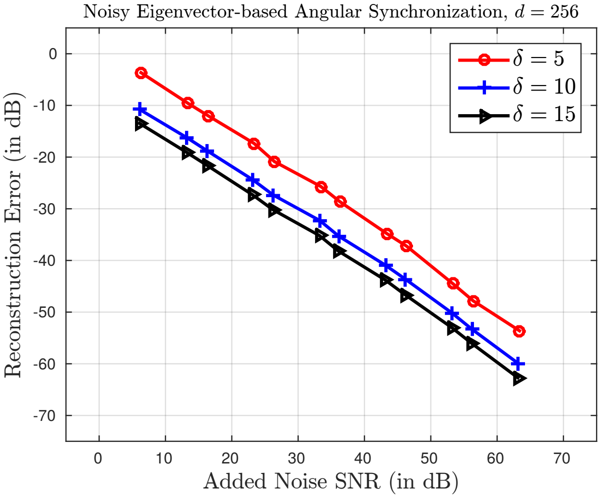
$$(Y')_{j,k} = (X')_{j,k} + \underbrace{\eta_{j,k}}_{\mathcal{CN}(0,\mathbb{I}_d)} = \begin{cases} (\mathbf{x}\mathbf{x}^*)_{j,k} + \eta_{j,k} & \text{if } |j - k \bmod d| < \delta \\ 0 & \text{otherwise.} \end{cases}$$

Noisy Angular Synchronization



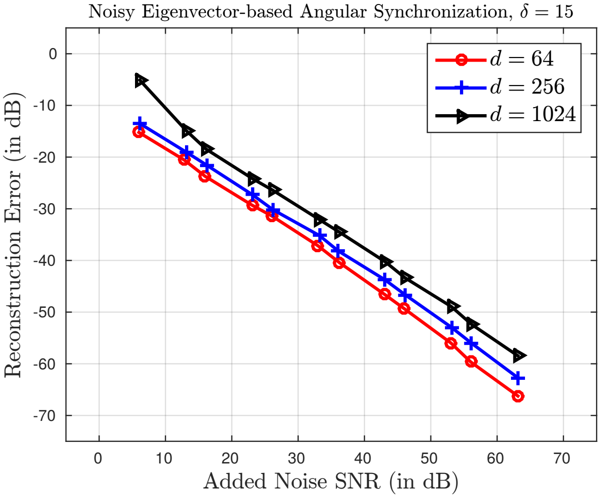
$$\text{Added Noise SNR (dB)} = 10 \log_{10} \left(\frac{\|X'\|_F^2}{\|Y' - X'\|_F^2} \right)$$

Noisy Angular Synchronization



$$\text{Reconstruction Error (dB)} = 10\log_{10}\left(\frac{\min_{\theta\in[0,2\pi)} \|\tilde{\mathbf{y}} - \mathrm{e}^{\mathrm{i}\theta}\tilde{\mathbf{x}}\|_2^2}{\|\tilde{\mathbf{x}}\|_2^2}\right)$$

Noisy Angular Synchronization



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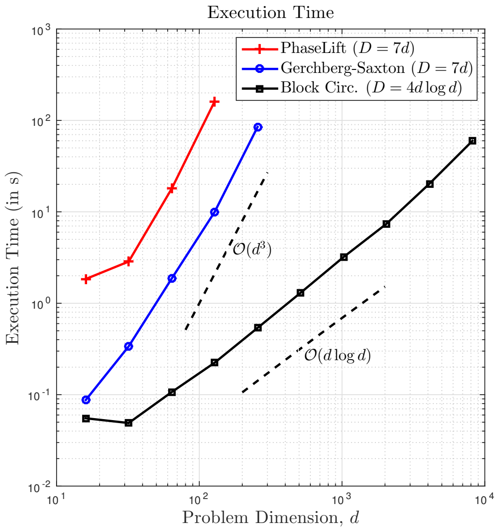
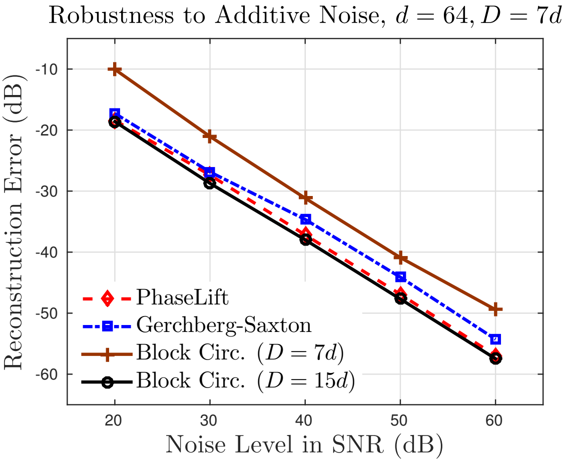
Application to Phase Retrieval: Efficiency



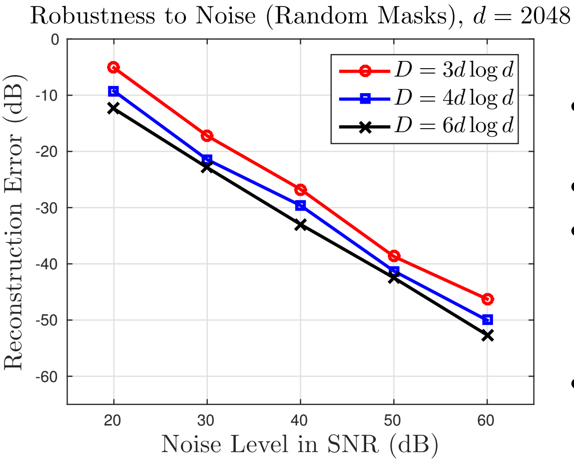
- iid Complex Gaussian test signal
- Block-Circulant Solve + Eigenvector-based angular synchronization
- $4d \log d$ measurements

Application to Phase Retrieval: Robustness



- iid complex Gaussian signal
- $d = 64$
- $7d$ measurements
- Deterministic (windowed Fourier-like) measurements

Application to Phase Retrieval: Robustness



- iid complex Gaussian signal
- $d = 2048$
- Not feasible with SDP-based methods such as PhaseLift on a laptop in Matlab
- Random measurements

Concluding Remarks

- BlockPR allows for essentially linear-time robust phase retrieval from block-circulant measurement constructions.
- Deterministic measurements for flat vectors.
- Eigenvector-based angular synchronization works when we have structured and highly incomplete phase difference measurements.

Coming Soon. . .

- Sublinear-time compressive phase retrieval
- Extensions to 2D and Ptychography

Publications/Preprints/Code

Block-Circulant Measurements + Greedy Angular Synchronization

Mark Iwen, A. Viswanathan and Yang Wang. "Fast Phase Retrieval for High-Dimensions." arXiv:1501.02377, 2015.

This talk (Eigenvector-based Angular Synchronization)

A. Viswanathan and Mark Iwen. "Fast Angular Synchronization for Phase Retrieval via Incomplete Information." Proc. SPIE 9597.

Code: https://bitbucket.org/charms/blockpr

Extensions to Compressive Phase Retrieval:

M. Iwen, A. Viswanathan, and Y. Wang. "Robust Sparse Phase Retrieval Made Easy." (in press) ACHA, 2015. arXiv:1410.5295

Code: https://bitbucket.org/charms/sparsepr

Coming soon: Sublinear-time compressive phase retrieval

Questions?