# Robust and Fast Phase Retrieval from Local Correlation Measurements


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Current and Previous Collaborators

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# The Phase Retrieval Problem

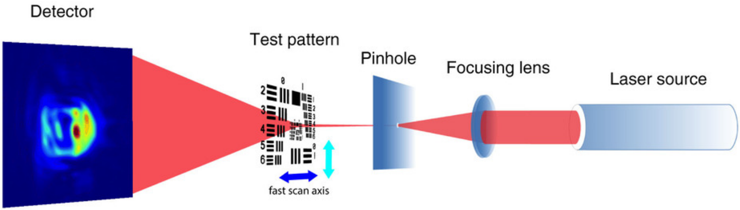
$$\text{find}^1 \quad \mathbf{x} \in \mathbb{C}^d \quad \text{given} \quad y_i = |\langle \mathbf{a}_i, \mathbf{x} \rangle|^2 + \eta_i \qquad i \in [D],$$

where

- $y_i \in \mathbb{R}$ denotes the phaseless (or magnitude-only) measurements ($D$ measurements acquired),
- $\mathbf{a}_i \in \mathbb{C}^d$ are known (by design or estimation) measurement vectors, and
- $\eta_i \in \mathbb{R}$ is measurement noise.

[1] (upto a global phase offset)

# Motivating Applications




From Huang, Xiaojing, et al. "Fly-scan ptychography." *Scientific Reports* 5 (2015).

The Phase Retrieval problem arises in many molecular imaging modalities, including

- X-ray crystallography
- Ptychography

Other applications can be found in optics, astronomy and speech processing.

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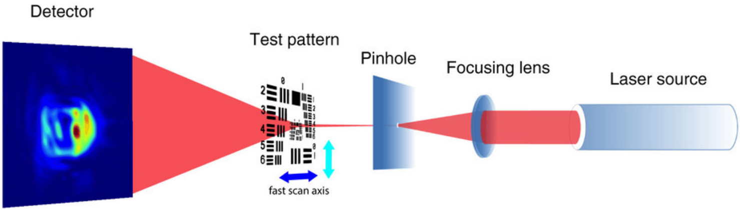


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The Phase Retrieval problem arises in many molecular imaging modalities, including

- X-ray crystallography
- Ptychography – this (local measurements) will be our main focus

Other applications can be found in optics, astronomy and speech processing.

# Existing Computational Approaches

- Alternating projection methods
  [Fienup, 1978], [Marchesini et al., 2006], [Fannjiang, Liao, 2012]
  and many others...
- Methods based on semidefinite programming
  PhaseLift [Candes et al., 2012], PhaseCut [Waldspurger et al., 2012], ...
- Others
  - Frame-theoretic, graph based algorithms [Alexeev et al., 2014]
  - (Stochastic) gradient descent [Candes et al., 2014]

...and variants for sparse and/or structured signal models.

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- Others
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Most methods (with recovery guarantees) require global, random measurement constructions.

# Today...

- We discuss a recently introduced essentially linear-time robust phase retrieval algorithm based on (deterministic[2]) local correlation measurement constructions.

- We provide rigorous theoretical recovery guarantees and present numerical results showing the accuracy, efficiency and robustness of the method.

[2]for a large class of real-world signals

# Outline

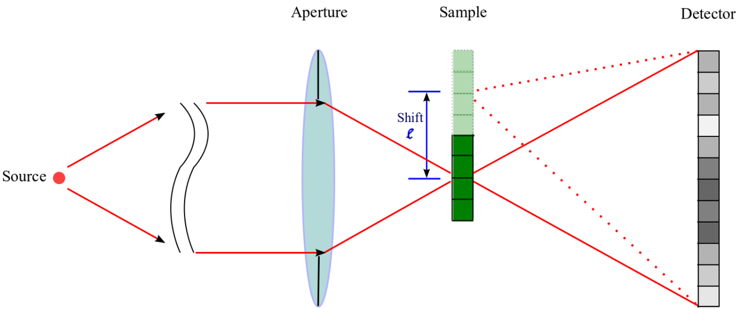
1 The Phase Retrieval Problem

2 BlockPR: Fast Phase Retrieval from Local Correlation Measurements
  - Measurement Constructions
  - Solving for Phase Differences
  - Angular Synchronization

3 Theoretical Guarantees

4 Numerical Simulations

# Measurement Constructions




Adapted from Huang et al. "Fly-scan ptychography." *Scientific Reports* 5 (2015).

- We consider measurements motivated by **Ptychographic** molecular imaging.
- Measurements are **local**; the full reconstruction is obtained by imaging *shifts* of the specimen.

# Our Model

Each $\mathbf{a}_i$ is a shift of a locally-supported vector

$$\mathbf{m}_j \in \mathbb{C}^d, \qquad \mathsf{supp}(\mathbf{m}_j) = [\delta] \subset [d], \qquad j = 1, \ldots, K$$

Define the discrete circular shift operator

$$S_\ell : \mathbb{C}^d \to \mathbb{C}^d, \qquad \text{with} \qquad (S_\ell \mathbf{x})_j = x_{\ell+j}.$$

Our measurements are then

$$(\mathbf{y}_\ell)_j = |\langle \mathbf{x}, S_\ell^* \mathbf{m}_j \rangle|^2 + \eta_{j\ell}, \quad (j, \ell) \in [K] \times P, \quad P \subset \{0, \ldots, d-1\}$$

We will consider $K \approx 2\delta - 1$ and $P = [d]_0 := \{0, \ldots, d-1\}$.

# What are we Measuring?

Our squared magnitude measurements can be written in terms of a lifted set of variables

$$|\langle \mathbf{x}, S_\ell^* \mathbf{m}_j \rangle|^2 = \langle \mathbf{x}\mathbf{x}^*, S_\ell^* \mathbf{m}_j \mathbf{m}_j^* S_\ell \rangle$$

Figure: Left and center: $d \times d$ matrices where the shaded $\delta \times \delta$ blocks are the support of $S_\ell^* m_j m_j^* S_\ell, \ell = 1, 2$.
Right: shaded region is the union of the supports of $S_\ell^* m_j m_j^* S_\ell, \ell \in [d]_0$.

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$$|\langle \mathbf{x}, S_\ell^* \mathbf{m}_j \rangle|^2 = \langle \mathbf{x}\mathbf{x}^*, S_\ell^* \mathbf{m}_j \mathbf{m}_j^* S_\ell \rangle$$

The only entries of $\mathbf{x}\mathbf{x}^*$ that we can hope to recover (by linear inversion) are those supported on a (circulant) band.

Defining the restriction operator $T_k : \mathbb{C}^{d\times d} \to \mathbb{C}^{d\times d}$ given by

$$T_k(G)_{ij} = \begin{cases} G_{ij}, & |i-j| \mod d < k \\ 0, & \text{otherwise.} \end{cases}$$

$$|\langle \mathbf{x}, S_\ell^* \mathbf{m}_j \rangle|^2 = \langle \mathbf{x}\mathbf{x}^*, S_\ell^* \mathbf{m}_j \mathbf{m}_j^* S_\ell \rangle = \langle T_\delta(\mathbf{x}\mathbf{x}^*), S_\ell^* \mathbf{m}_j \mathbf{m}_j^* S_\ell \rangle$$

# The Bottom Line

If Span$\{S_\ell^* \mathbf{m}_j \mathbf{m}_j^* S_\ell\}_{\ell,j} = T_\delta(\mathbb{C}^{d\times d})$, we can recover $T_\delta(\mathbf{x}\mathbf{x}^*)$.

What does this give us?

1. Diagonal entries of $T_\delta(\mathbf{x}\mathbf{x}^*)$ are $|x_i|^2$.
2. Leading eigenvector of $T_\delta\left(\frac{\mathbf{x}\mathbf{x}^*}{|\mathbf{x}\mathbf{x}^*|}\right)$ (appropriately normalized) is the vector of phases of $\mathbf{x}$.

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Why?

$$\begin{aligned} T_\delta\left(\frac{\mathbf{x}\mathbf{x}^*}{|\mathbf{x}\mathbf{x}^*|}\right) &= \mathrm{diag}\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right) T_\delta(\mathbb{1}\mathbb{1}^*)\mathrm{diag}\left(\frac{\mathbf{x}^*}{|\mathbf{x}|}\right) \\ &= \mathrm{diag}\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right) F\Lambda F^* \mathrm{diag}\left(\frac{\mathbf{x}^*}{|\mathbf{x}|}\right) \end{aligned}$$

# The Bottom Line

If $\text{Span}\{S_\ell^* \mathbf{m}_j \mathbf{m}_j^* S_\ell\}_{\ell,j} = T_\delta(\mathbb{C}^{d\times d})$, we can recover $T_\delta(\mathbf{x}\mathbf{x}^*)$.

What does this give us?

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Why?

$$T_\delta\left(\frac{\mathbf{x}\mathbf{x}^*}{|\mathbf{x}\mathbf{x}^*|}\right) = \text{diag}\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right) T_\delta(\mathbb{1}\mathbb{1}^*)\text{diag}\left(\frac{\mathbf{x}^*}{|\mathbf{x}|}\right)$$
$$= \text{diag}\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right) F\Lambda F^*\text{diag}\left(\frac{\mathbf{x}^*}{|\mathbf{x}|}\right)$$

Hence, leading eigenvector of $T_\delta(\mathbf{x}\mathbf{x}^*)$ gives the phase vector $\frac{\mathbf{x}}{|\mathbf{x}|}$ (upto a global phase factor).

# An Example... $(d = 4, \delta = 2)$

Our *local correlation measurements* can be written as

$$(\mathbf{y}_\ell)_i = \left| \sum_{k=1}^{\delta=2} (\mathbf{m}_\ell)_k \cdot x_{i+k-1} \right|^2, \qquad (\ell, i) \in \{1,2,3\} \times \{1,2,3,4\}$$

$$= \sum_{j,k=1}^{\delta} (\mathbf{m}_\ell)_j \, (\mathbf{m}_\ell)_k^* \, x_{i+j-1} \, x_{i+k-1}^* := \sum_{j,k=1}^{\delta} (\mathbf{m}_\ell)_{j,k} \, x_{i+j-1} \, x_{i+k-1}^*.$$

This is a linear system for the phase differences $\{x_j x_k^*\}$!

*Note:* the masks $\mathbf{m}_{\{1,2,3\}}$ are known - either by design or through calibration.

# Solving for Phase Differences

Writing out the correlation sum, we obtain the linear system

$$M'\mathbf{z} = \widetilde{\mathbf{b}},$$

where

$$\mathbf{z} = \left[ |x_1|^2 \;\; x_1x_2^* \;\; x_2x_1^* \;\; |x_2|^2 \;\; x_2x_3^* \;\; x_3x_2^* \;\; |x_3|^2 \;\; x_3x_4^* \;\; x_4x_3^* \;\; |x_4|^2 \;\; x_4x_1^* \;\; x_1x_4^* \right]^T,$$

$$\widetilde{\mathbf{b}} = \left[ (y_1)_1 \;\; (y_2)_1 \;\; (y_3)_1 \;\; (y_1)_2 \;\; (y_2)_2 \;\; (y_3)_2 \;\; (y_1)_3 \;\; (y_2)_3 \;\; (y_3)_3 \;\; (y_1)_4 \;\; (y_2)_4 \;\; (y_3)_4 \right]^T,$$

$$M' = \begin{bmatrix}
(\mathbf{m}_1)_{1,1} & (\mathbf{m}_1)_{1,2} & (\mathbf{m}_1)_{2,1} & (\mathbf{m}_1)_{2,2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(\mathbf{m}_2)_{1,1} & (\mathbf{m}_2)_{1,2} & (\mathbf{m}_2)_{2,1} & (\mathbf{m}_2)_{2,2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(\mathbf{m}_3)_{1,1} & (\mathbf{m}_3)_{1,2} & (\mathbf{m}_3)_{2,1} & (\mathbf{m}_3)_{2,2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & (\mathbf{m}_1)_{1,1} & (\mathbf{m}_1)_{1,2} & (\mathbf{m}_1)_{2,1} & (\mathbf{m}_1)_{2,2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & (\mathbf{m}_2)_{1,1} & (\mathbf{m}_2)_{1,2} & (\mathbf{m}_2)_{2,1} & (\mathbf{m}_2)_{2,2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & (\mathbf{m}_3)_{1,1} & (\mathbf{m}_3)_{1,2} & (\mathbf{m}_3)_{2,1} & (\mathbf{m}_3)_{2,2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & (\mathbf{m}_1)_{1,1} & (\mathbf{m}_1)_{1,2} & (\mathbf{m}_1)_{2,1} & (\mathbf{m}_1)_{2,2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & (\mathbf{m}_2)_{1,1} & (\mathbf{m}_2)_{1,2} & (\mathbf{m}_2)_{2,1} & (\mathbf{m}_2)_{2,2} & 0 & 0 \\
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\end{bmatrix}$$

# Back to our Example . . .

$$\left[|x_1|^2 \; x_1x_2^* \; x_2x_1^* \; |x_2|^2 \; x_2x_3^* \; x_3x_2^* \; |x_3|^2 \; x_3x_4^* \; x_4x_3^* \; |x_4|^2 \; x_4x_1^* \; x_1x_4^*\right]^T$$

$\downarrow$ (re-arrange)

$$\begin{bmatrix} |x_1|^2 & x_1x_2^* & 0 & x_1x_4^* \\ x_2x_1^* & |x_2|^2 & x_2x_3^* & 0 \\ 0 & x_3x_2^* & |x_3|^2 & x_3x_4^* \\ x_4x_1^* & 0 & x_4x_3^* & |x_4|^2 \end{bmatrix}$$ ($2\delta - 1$ entries in band)

$\downarrow$ (normalize)

$$\begin{bmatrix} 1 & e^{\mathbb{i}(\phi_1-\phi_2)} & 0 & e^{\mathbb{i}(\phi_1-\phi_4)} \\ e^{\mathbb{i}(\phi_2-\phi_1)} & 1 & e^{\mathbb{i}(\phi_2-\phi_3)} & 0 \\ 0 & e^{\mathbb{i}(\phi_3-\phi_2)} & 1 & e^{\mathbb{i}(\phi_3-\phi_4)} \\ e^{\mathbb{i}(\phi_4-\phi_1)} & 0 & e^{\mathbb{i}(\phi_4-\phi_3)} & 1 \end{bmatrix}$$

$\downarrow$ (angular synchronization)

$$\phi_1, \phi_2, \phi_3, \phi_4$$

(Signal Reconstruction) $\left[|x_1|e^{\mathbb{i}\,\phi_1} \quad |x_2|e^{\mathbb{i}\,\phi_2} \quad |x_3|e^{\mathbb{i}\,\phi_3} \quad |x_4|e^{\mathbb{i}\,\phi_4}\right]^T$

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# BlockPR – Algorithmic Components

1. **Local Measurements**: Each measurement provides information about some *local* region of $\mathbf{x}$.

2. **Local Lifting**: Use compactly supported masks and correlation measurements to obtain phase difference estimates.

$$|\texttt{Corr}(\mathbf{m}_i, \mathbf{x})|^2 \xrightarrow[\text{linear system}]{\text{solve}} \{x_j x_k^*\}_{|j-k \mod d < \delta|}$$

   - $\mathbf{m}_i$ is a mask or window function with $\delta$ non-zero entries.
   - $x_j x_k^*$ provides (scaled) phase difference between $x_j$ and $x_k$.

3. **Angular Synchronization**: Use the phase differences to obtain the phases of the unknown signal.

$$\{x_j x_k^*\}_{|j-k \mod d < \delta|} \xrightarrow[\text{synchronization}]{\text{angular}} \{x_j\}_{j=1}^d$$

*Computational Complexity*: $\mathcal{O}(\delta^2 \cdot d \log d + d \cdot \delta^3)$

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# Well-Conditioned Linear Systems

**Theorem (Iwen, V., Wang 2015)**

*Choose entries of the measurement mask* $(\mathbf{m}_i)$ *as follows:*

$$(\mathbf{m}_i)_\ell = \begin{cases} \frac{\mathrm{e}^{-\ell/a}}{\sqrt[4]{2\delta-1}} \cdot \mathrm{e}^{\frac{2\pi \mathrm{i} \cdot i \cdot \ell}{2\delta-1}}, & \ell \leq \delta \\ 0, & \ell > \delta \end{cases}, \qquad \begin{array}{l} a := \max\left\{4, \frac{\delta-1}{2}\right\}, \\ i = 1, 2, \ldots, N. \end{array}$$

*Then, the resulting system matrix for the phase differences,* $M'$*, has condition number*

$$\kappa(M') < \max\left\{144\mathrm{e}^2, \frac{9\mathrm{e}^2}{4} \cdot (\delta - 1)^2\right\}.$$

- Deterministic (windowed DFT-type) measurement masks!
- $\delta$ is typically chosen to be $c \log_2 d$ with $c$ small ($2$–$3$).
- Extensions: oversampling, random masks . . . .

# Well-Conditioned Linear Systems

Mask Construction II (Iwen, Preskitt, Saab, V. 2016)

Choose entries of the measurement mask $(\mathbf{m}_i)$ as follows:
For $i = 1, 2, \ldots, \delta - 1$

$$\begin{aligned} \mathbf{m}_1 &= \mathbf{e}_1 \\ \mathbf{m}_{2i} &= \mathbf{e}_1 + \mathbf{e}_{i+1} \\ \mathbf{m}_{2i+1} &= \mathbf{e}_1 - \mathbb{i}\mathbf{e}_{i+1} \end{aligned}$$

Then, the resulting system matrix for the phase differences, $M'$, has condition number

$$\kappa(M') < c\delta.$$

# Recovery Guarantee

**Theorem (Iwen, Preskitt, Saab, V. 2016)**

*Let* $x_{\min} := \min_j |x_j|$ *be the smallest magnitude of any entry in* $\mathbf{x}$*. Then, the estimate* $\mathbf{z}$ *produced by the proposed BlockPR Algorithm satisfies*

$$\min_{\theta \in [0, 2\pi]} \left\| \mathbf{x} - \mathbb{e}^{\mathbb{i}\theta} \mathbf{z} \right\|_2 \leq C \left( \frac{\|\mathbf{x}\|_\infty}{x_{\min}^2} \right) \left( \frac{d}{\delta} \right)^2 \kappa \|\eta\|_2 + C d^{\frac{1}{4}} \sqrt{\kappa \|\eta\|_2},$$

*where* $C \in \mathbb{R}^+$ *is an absolute universal constant.*

- This result yields a *deterministic* recovery result for any signal $\mathbf{x}$ which contains no zero entries.
- A randomized result can be derived for arbitrary $\mathbf{x}$ by right multiplying the signal $\mathbf{x}$ with a random "flattening" matrix.

# Main Elements of the Proof

1. Well-conditioned measurements:
   - Linear system for the lifted variables is block-circulant
   - Bound condition number of each block to find $\kappa$.
2. (Reconstruction error) $\approx$ (Phase error) $+$ (Magnitude error)
   - Magnitude error (second term in error guarantee) – follows from error in inverting linear system for lifted variables
   - Phase error (first term in error guarantee) – evaluate eigenvalue gap $+$ Cheeger inequality of [Bandeira et al. 2013] $+$ adaptation of proof method from [Alexeev et al. 2014]

*Note:* Bound not optimized; for example, magnitude estimation can be improved!

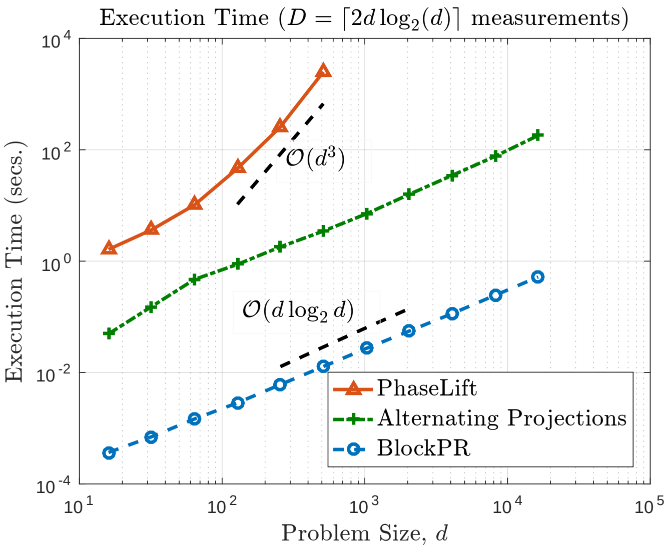
# Outline

1. The Phase Retrieval Problem
2. BlockPR: Fast Phase Retrieval from Local Correlation Measurements
   - Measurement Constructions
   - Solving for Phase Differences
   - Angular Synchronization
3. Theoretical Guarantees
4. Numerical Simulations

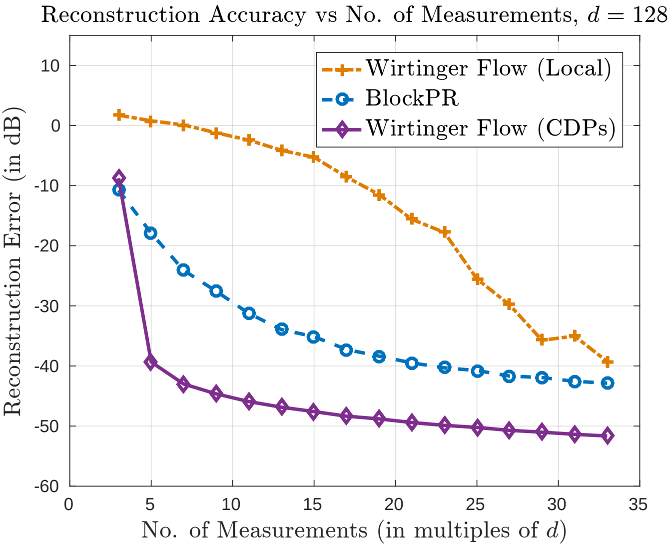
# Simulation Parameters

- *Signal*: complex Gaussian test signals
- *Noise*: iid additive Gaussian noise
- *Measurements*: local, deterministic (complex, constructed using canonical bases – from §2.1)
- data points obtained by averaging over $100$ trials
- simulations performed in Matlab on a laptop computer
- for *PhaseLift* – implemented in CVX as a trace regularized least squares problem
- for *Alternating Projections* – random complex Gaussian initial guess, max. iterations $= 10,000$

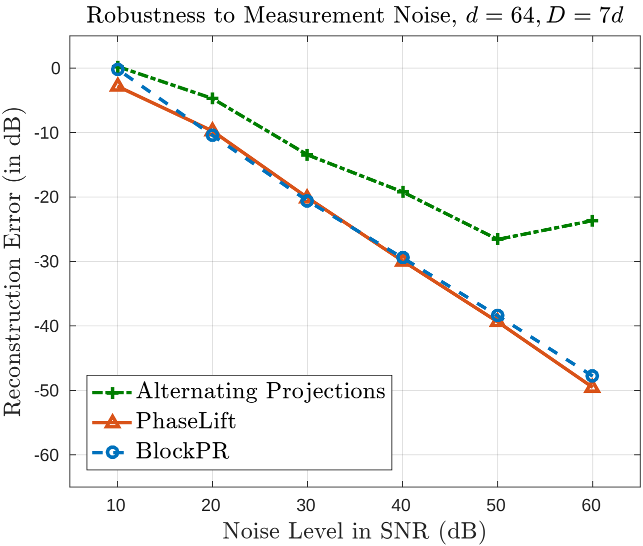
# Efficiency – FFT–time phase retrieval



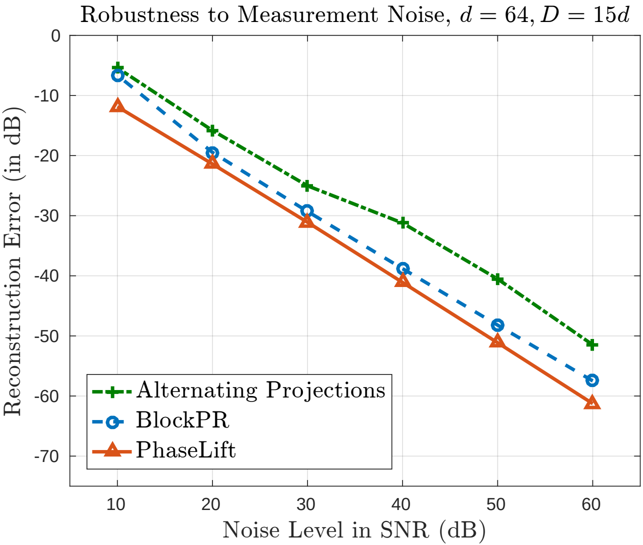
# Local vs Global Measurements



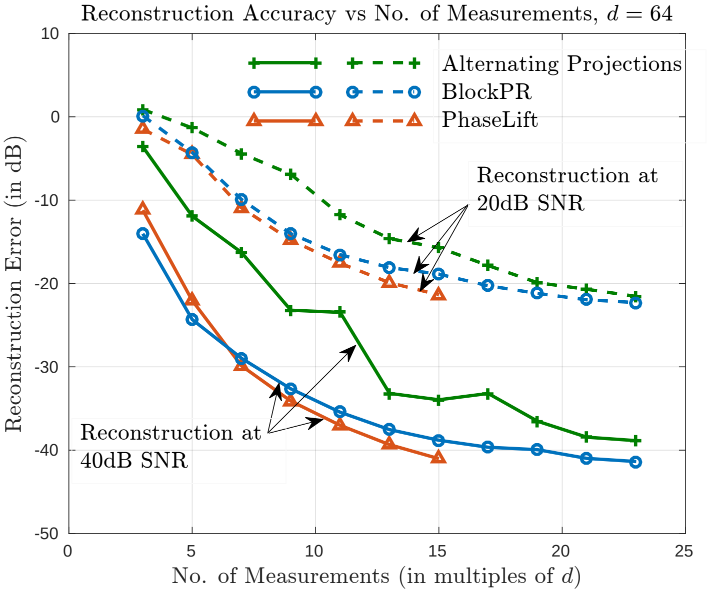
# Robustness to Measurement Errors



# Robustness to Measurement Errors



# Reconstruction Error vs. No. of Measurements



# In Summary...

- BlockPR allows for essentially linear-time robust phase retrieval from local correlation measurement constructions.
- Deterministic measurements for flat vectors.
- **Global** *robust* recovery guarantee for phase retrieval from local correlation (ptychographic) measurements.
- *Improved* robustness from **eigenvector**-based angular synchronization.

Current and Future Directions

- (Sublinear-time) compressive phase retrieval
- Extensions to 2D and Ptychographic data sets
- Continuous/infinite dimensional formulation (coming SAMPTA 2017...?)

# Publications/Preprints/Code (see https://math.msu.edu/∼aditya)

This Talk

M. Iwen, B. Preskitt, R. Saab and A. Viswanathan. "Phase Retrieval from Local Measurements: Improved Robustness via Eigenvector-Based Angular Synchronization." arXiv:1612.01182, 2016.

Related Work

M. Iwen, A. Viswanathan, and Y. Wang. "Fast Phase Retrieval from Local Correlation Measurements." SIAM J. Imag. Sci., Vol. 9, Issue 4, pp. 1655–1688, Oct. 2016.

M. Iwen, A. Viswanathan, and Y. Wang. "Robust Sparse Phase Retrieval Made Easy." Vol. 42, pp. 135–142, 2017.

Code:
https://bitbucket.org/charms/blockpr (this talk)
https://bitbucket.org/charms/sparsepr (sparse phase retrieval)

# Questions?