Robust and Fast Phase Retrieval from Local Correlation Measurements

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Current and Previous Collaborators

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The Phase Retrieval Problem

\[ \text{find}^1 \ \mathbf{x} \in \mathbb{C}^d \ \text{given} \ \ y_i = |\langle \mathbf{a}_i, \mathbf{x} \rangle|^2 + \eta_i \quad i \in [D], \]

where

- \( y_i \in \mathbb{R} \) denotes the phaseless (or magnitude-only) measurements (\( D \) measurements acquired),
- \( \mathbf{a}_i \in \mathbb{C}^d \) are known (by design or estimation) measurement vectors, and
- \( \eta_i \in \mathbb{R} \) is measurement noise.

\(^1\text{(upto a global phase offset)}\)
Motivating Applications

The Phase Retrieval problem arises in many molecular imaging modalities, including

- X-ray crystallography
- Ptychography

Other applications can be found in optics, astronomy and speech processing.

The Phase Retrieval problem arises in many molecular imaging modalities, including

- X-ray crystallography
- Ptychography – this (local measurements) will be our main focus

Other applications can be found in optics, astronomy and speech processing.
Existing Computational Approaches

• Alternating projection methods
  [Fienup, 1978], [Marchesini et al., 2006], [Fannjiang, Liao, 2012] and many others...

• Methods based on semidefinite programming
  PhaseLift [Candes et al., 2012], PhaseCut [Waldspurger et al., 2012], ...

• Others
  • Frame-theoretic, graph based algorithms [Alexeev et al., 2014]
  • (Stochastic) gradient descent [Candes et al., 2014]

... and variants for sparse and/or structured signal models.
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- Others
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Most methods (with recovery guarantees) require global, random measurement constructions.
Today...

- We discuss a recently introduced essentially linear-time robust phase retrieval algorithm based on (deterministic\(^2\)) local correlation measurement constructions.

- We provide rigorous theoretical recovery guarantees and present numerical results showing the accuracy, efficiency and robustness of the method.

\(^2\)for a large class of real-world signals
Outline

1. The Phase Retrieval Problem

2. BlockPR: Fast Phase Retrieval from Local Correlation Measurements
   - Measurement Constructions
   - Solving for Phase Differences
   - Angular Synchronization

3. Theoretical Guarantees

4. Numerical Simulations
We consider measurements motivated by Ptychographic molecular imaging.

Measurements are local; the full reconstruction is obtained by imaging shifts of the specimen.
Our Model

Each $a_i$ is a shift of a locally-supported vector

$$m_j \in \mathbb{C}^d, \quad \text{supp}(m_j) = [\delta] \subset [d], \quad j = 1, \ldots, K$$

Define the discrete circular shift operator

$$S_\ell : \mathbb{C}^d \rightarrow \mathbb{C}^d, \quad \text{with} \quad (S_\ell x)_j = x_{\ell+j}.$$  

Our measurements are then

$$(y_\ell)_j = |\langle x, S_\ell^* m_j \rangle|^2 + \eta_{j\ell}, \quad (j, \ell) \in [K] \times P, \quad P \subset \{0, \ldots, d-1\}$$

We will consider $K \approx 2\delta - 1$ and $P = [d]_0 := \{0, \ldots, d - 1\}$. 
What are we Measuring?

Our squared magnitude measurements can be written in terms of a lifted set of variables

\[ |\langle x, S^*_\ell m_j \rangle|^2 = \langle xx^*, S^*_\ell m_j m^*_j S_\ell \rangle \]

Figure: Left and center: $d \times d$ matrices where the shaded $\delta \times \delta$ blocks are the support of $S^*_\ell m_j m^*_j S_\ell$, $\ell = 1, 2$.
Right: shaded region is the union of the supports of $S^*_\ell m_j m^*_j S_\ell$, $\ell \in [d]_0$. 
What are we Measuring?

Our squared magnitude measurements can be written in terms of a lifted set of variables

$$|\langle x, S^*_\ell m_j \rangle|^2 = \langle xx^*, S^*_\ell m_j m^*_j S_\ell \rangle$$

The only entries of $xx^*$ that we can hope to recover (by linear inversion) are those supported on a (circulant) band.

Defining the restriction operator $T_k : \mathbb{C}^{d \times d} \rightarrow \mathbb{C}^{d \times d}$ given by

$$T_k(G)_{ij} = \begin{cases} G_{ij}, & |i - j| \text{ mod } d < k \\ 0, & \text{otherwise.} \end{cases}$$

$$|\langle x, S^*_\ell m_j \rangle|^2 = \langle xx^*, S^*_\ell m_j m^*_j S_\ell \rangle = \langle T_\delta(xx^*), S^*_\ell m_j m^*_j S_\ell \rangle$$
If \( \text{Span}\{S^*_\ell m_j m^*_j S_{\ell}\}_{\ell,j} = T_\delta(\mathbb{C}^{d \times d}) \), we can recover \( T_\delta(xx^*) \).

What does this give us?

1. Diagonal entries of \( T_\delta(xx^*) \) are \( |x_i|^2 \).
2. Leading eigenvector of \( T_\delta \left( \frac{xx^*}{|xx^*|} \right) \) (appropriately normalized) is the vector of phases of \( x \).
If \( \text{Span}\{S_{\ell}^*m_j m_j^* S_{\ell}\}_{\ell,j} = T_\delta(\mathbb{C}^{d\times d}) \), we can recover \( T_\delta(xx^*) \).

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The Bottom Line

If \( \text{Span}\{S^*_\ell m^*_j m^*_j S^*_\ell\}_{\ell,j} = T_\delta(\mathbb{C}^{d \times d}) \), we can recover \( T_\delta(\mathbf{x}\mathbf{x}^*) \).

What does this give us?

1. Diagonal entries of \( T_\delta(\mathbf{x}\mathbf{x}^*) \) are \( |x_i|^2 \).
2. Leading eigenvector of \( T_\delta\left(\frac{\mathbf{x}\mathbf{x}^*}{|\mathbf{x}\mathbf{x}^*|}\right) \) (appropriately normalized) is the vector of phases of \( \mathbf{x} \).

Why?

\[
T_\delta\left(\frac{\mathbf{x}\mathbf{x}^*}{|\mathbf{x}\mathbf{x}^*|}\right) = \text{diag} \left(\frac{\mathbf{x}}{|\mathbf{x}|}\right) T_\delta(\mathbf{1}\mathbf{1}^*) \text{diag} \left(\frac{\mathbf{x}^*}{|\mathbf{x}|}\right) \\
= \text{diag} \left(\frac{\mathbf{x}}{|\mathbf{x}|}\right) F\Lambda F^* \text{diag} \left(\frac{\mathbf{x}^*}{|\mathbf{x}|}\right)
\]
If \( \text{Span}\{S_{\ell}^*m_jm_j^*S_{\ell}\}_{\ell,j} = T_\delta(\mathbb{C}^{d\times d}) \), we can recover \( T_\delta(xx^*) \).

What does this give us?

1. Diagonal entries of \( T_\delta(xx^*) \) are \( |x_i|^2 \).
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\[
T_\delta\left(\frac{xx^*}{|xx^*|}\right) = \text{diag}\left(\frac{x}{|x|}\right) T_\delta(11^*)\text{diag}\left(\frac{x^*}{|x|}\right)
\]

\[
= \text{diag}\left(\frac{x}{|x|}\right) F\Lambda F^*\text{diag}\left(\frac{x^*}{|x|}\right)
\]

Hence, leading eigenvector of \( T_\delta(xx^*) \) gives the phase vector \( \frac{x}{|x|} \) (upto a global phase factor).
An Example... \((d = 4, \delta = 2)\)

Our *local correlation measurements* can be written as

\[
(y_{\ell})_i = \left| \sum_{k=1}^{\delta=2} (m_{\ell})_k \cdot x_{i+k-1} \right|^2, \quad (\ell, i) \in \{1, 2, 3\} \times \{1, 2, 3, 4\}
\]

\[
= \sum_{j, k=1}^{\delta} (m_{\ell})_j (m_{\ell})_k^* x_{i+j-1} x_{i+k-1} := \sum_{j, k=1}^{\delta} (m_{\ell})_{j,k} x_{i+j-1} x_{i+k-1}^*.
\]

This is a *linear* system for the phase differences \(\{x_j x_k^*\}\)!

*Note:* the masks \(m_{\{1,2,3\}}\) are known - either by design or through calibration.
Solving for Phase Differences

Writing out the correlation sum, we obtain the linear system

\[ M' \mathbf{z} = \tilde{\mathbf{b}}, \]

where

\[ \mathbf{z} = \begin{bmatrix} |x_1|^2 & x_1x_2^* & x_2x_3^* & |x_2|^2 & x_2x_3^* & x_3x_4^* & |x_3|^2 & x_3x_4^* & |x_4|^2 & x_4x_1^* & x_1x_4^* \end{bmatrix}^T, \]

\[ \tilde{\mathbf{b}} = \begin{bmatrix} (y_1)_1 & (y_2)_1 & (y_3)_1 & (y_1)_2 & (y_2)_2 & (y_3)_2 & (y_1)_3 & (y_2)_3 & (y_3)_3 & (y_1)_4 & (y_2)_4 & (y_3)_4 \end{bmatrix}^T, \]

\[ M' = \begin{bmatrix} (m_1)_{1,1} & (m_1)_{1,2} & (m_1)_{2,1} & (m_1)_{2,2} & 0 & 0; & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \(m_2)_{1,1} & (m_2)_{1,2} & (m_2)_{2,1} & (m_2)_{2,2} & 0 & 0; & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \(m_3)_{1,1} & (m_3)_{1,2} & (m_3)_{2,1} & (m_3)_{2,2} & 0 & 0; & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]
Back to our Example . . .

$$\begin{bmatrix} |x_1|^2 & x_1x_2^* & x_2x_1^* & |x_2|^2 & x_2x_3^* & x_3x_2^* & |x_3|^2 & x_3x_4^* & x_4x_3^* & |x_4|^2 & x_4x_1^* & x_1x_4^* \end{bmatrix}^T$$

(re-arrange)

$$\begin{bmatrix} |x_1|^2 & x_1x_2^* & 0 & x_1x_4^* \\ x_2x_1^* & |x_2|^2 & x_2x_3^* & 0 \\ 0 & x_3x_2^* & |x_3|^2 & x_3x_4^* \\ x_4x_1^* & 0 & x_4x_3^* & |x_4|^2 \end{bmatrix}$$

(2δ − 1 entries in band)

(normalize)

$$\begin{bmatrix} 1 & e^{i(\phi_1-\phi_2)} & 0 & e^{i(\phi_1-\phi_4)} \\ e^{i(\phi_2-\phi_1)} & 1 & e^{i(\phi_2-\phi_3)} & 0 \\ 0 & e^{i(\phi_3-\phi_2)} & 1 & e^{i(\phi_3-\phi_4)} \\ e^{i(\phi_4-\phi_1)} & 0 & e^{i(\phi_4-\phi_3)} & 1 \end{bmatrix}$$

(angular synchronization)

$$\phi_1, \phi_2, \phi_3, \phi_4$$

(Signal Reconstruction)

$$\begin{bmatrix} |x_1|e^{i\phi_1} & |x_2|e^{i\phi_2} & |x_3|e^{i\phi_3} & |x_4|e^{i\phi_4} \end{bmatrix}^T$$
Back to our Example . . .

\[
\begin{bmatrix}
|x_1|^2 & x_1x_2^* & x_2x_1^* & |x_2|^2 & x_2x_3^* & x_3x_2^* & |x_3|^2 & x_3x_4^* & x_4x_3^* & |x_4|^2 & x_4x_1^* & x_1x_4^*
\end{bmatrix}^T
\]

\[
\downarrow \text{(re-arrange)}
\]

\[
\begin{bmatrix}
|x_1|^2 & x_1x_2^* & 0 & x_1x_4^* \\
-x_2x_1^* & |x_2|^2 & x_2x_3^* & 0 \\
0 & -x_3x_2^* & |x_3|^2 & x_3x_4^* \\
x_4x_1^* & 0 & -x_4x_3^* & |x_4|^2
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\[
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1 & e^{i(\phi_1-\phi_2)} & 0 & e^{i(\phi_1-\phi_4)} \\
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\left[ |x_1|^2 \ x_1x_2^* \ x_2x_1^* \ |x_2|^2 \ x_2x_3^* \ x_3x_2^* \ |x_3|^2 \ x_3x_4^* \ x_4x_3^* \ |x_4|^2 \ x_4x_1^* \ x_1x_4^* \right]^T
\]

\[
\downarrow \text{(re-arrange)}
\]

\[
\begin{bmatrix}
|x_1|^2 & x_1x_2^* & 0 & x_1x_4^* \\
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BlockPR – Algorithmic Components

1 **Local Measurements**: Each measurement provides information about some local region of $x$.

2 **Local Lifting**: Use compactly supported masks and correlation measurements to obtain phase difference estimates.

\[
|\text{Corr}(m_i, x)|^2 \xrightarrow{\text{solve linear system}} \{x_j x_k^*\}_{|j-k| \mod d < \delta}
\]

- $m_i$ is a mask or window function with $\delta$ non-zero entries.
- $x_j x_k^*$ provides (scaled) phase difference between $x_j$ and $x_k$.

3 **Angular Synchronization**: Use the phase differences to obtain the phases of the unknown signal.

\[
\{x_j x_k^*\}_{|j-k| \mod d < \delta} \xrightarrow{\text{angular synchronization}} \{x_j\}_{j=1}^d
\]

**Computational Complexity**: $O(\delta^2 \cdot d \log d + d \cdot \delta^3)$
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Theorem (Iwen, V., Wang 2015)

Choose entries of the measurement mask \((m_i)\) as follows:

\[
(m_i)_\ell = \begin{cases} 
\frac{e^{-\ell/a}}{\sqrt{2\delta-1}} \cdot e^{\frac{2\pi i \cdot i \cdot \ell}{2\delta-1}}, & \ell \leq \delta \\
0, & \ell > \delta 
\end{cases}, \quad i = 1, 2, \ldots, N.
\]

Then, the resulting system matrix for the phase differences, \(M'\), has condition number

\[
\kappa(M') < \max \left\{ 144e^2, \frac{9e^2}{4} \cdot (\delta - 1)^2 \right\}.
\]

- **Deterministic** (windowed DFT-type) measurement masks!
- \(\delta\) is typically chosen to be \(c \log_2 d\) with \(c\) small (2–3).
- Extensions: oversampling, random masks . . . .
Well-Conditioned Linear Systems

**Mask Construction II** (Iwen, Preskitt, Saab, V. 2016)

Choose entries of the measurement mask \((m_i)\) as follows:
For \(i = 1, 2, \ldots, \delta - 1\)

\[
\begin{align*}
m_1 &= e_1 \\
m_{2i} &= e_1 + e_{i+1} \\
m_{2i+1} &= e_1 - ie_{i+1}
\end{align*}
\]

Then, the resulting system matrix for the phase differences, \(M'\), has condition number

\[\kappa(M') < c\delta.\]
Recovery Guarantee

**Theorem (Iwen, Preskitt, Saab, V. 2016)**

Let \( x_{\text{min}} := \min_j |x_j| \) be the smallest magnitude of any entry in \( x \). Then, the estimate \( z \) produced by the proposed BlockPR Algorithm satisfies

\[
\min_{\theta \in [0,2\pi]} \| x - e^{i\theta} z \|_2 \leq C \left( \frac{\|x\|_\infty}{x_{\text{min}}^2} \right) \left( \frac{d}{\delta} \right)^2 \kappa \|\eta\|_2 + C d^{1/4} \sqrt{\kappa \|\eta\|_2},
\]

where \( C \in \mathbb{R}^+ \) is an absolute universal constant.

- This result yields a deterministic recovery result for any signal \( x \) which contains no zero entries.

- A randomized result can be derived for arbitrary \( x \) by right multiplying the signal \( x \) with a random “flattening” matrix.
Main Elements of the Proof

1. Well-conditioned measurements:
   - Linear system for the lifted variables is block-circulant
   - Bound condition number of each block to find $\kappa$.

2. (Reconstruction error) $\approx$ (Phase error) + (Magnitude error)
   - Magnitude error (second term in error guarantee) – follows from error in inverting linear system for lifted variables

*Note*: Bound not optimized; for example, magnitude estimation can be improved!
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Simulation Parameters

- **Signal**: complex Gaussian test signals
- **Noise**: iid additive Gaussian noise
- **Measurements**: local, deterministic (complex, constructed using canonical bases – from §2.1)
- data points obtained by averaging over 100 trials
- simulations performed in Matlab on a laptop computer
- for **PhaseLift** – implemented in CVX as a trace regularized least squares problem
- for **Alternating Projections** – random complex Gaussian initial guess, max. iterations = 10,000
Efficiency — FFT–time phase retrieval

Execution Time ($D = \lceil 2d \log_2(d) \rceil$ measurements)
Local vs Global Measurements

Reconstruction Accuracy vs No. of Measurements, $d = 128$

- Wirtinger Flow (Local)
- BlockPR
- Wirtinger Flow (CDPs)
Robustness to Measurement Errors

Robustness to Measurement Noise, $d = 64$, $D = 7d$

- Alternating Projections
- PhaseLift
- BlockPR
Robustness to Measurement Errors

Robustness to Measurement Noise, $d = 64, D = 15d$

- Noise Level in SNR (dB)
- Reconstruction Error (in dB)
- Alternating Projections
- BlockPR
- PhaseLift
Reconstruction Error vs. No. of Measurements

Reconstruction Accuracy vs No. of Measurements, \( d = 64 \)

- Alternating Projections
- BlockPR
- PhaseLift

Reconstruction at 20dB SNR

Reconstruction at 40dB SNR
In Summary...

- BlockPR allows for essentially linear-time robust phase retrieval from local correlation measurement constructions.

- Deterministic measurements for flat vectors.

- **Global** robust recovery guarantee for phase retrieval from local correlation (ptychographic) measurements.

- Improved robustness from **eigenvector**-based angular synchronization.

Current and Future Directions

- (Sublinear-time) compressive phase retrieval
- Extensions to 2D and Ptychographic data sets
- Continuous/infinite dimensional formulation (coming SAMPTA 2017...?)
This Talk


Related Work


Code:
https://bitbucket.org/charms/blockpr (this talk)
https://bitbucket.org/charms/sparsepr (sparse phase retrieval)
Questions?