Reconstruction from Non-uniform Spectral Data

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Introduction

- We discuss the reconstruction of compactly supported functions from non-harmonic samples of their Fourier transform.
- This problem is of significance in applications such as MR imaging.

Let $f \in L^2(\mathbb{R})$ be compactly supported in $[-\pi, \pi)$. Given the samples $\langle f, e^{i\omega_k x}\rangle$, ω_k not necessarily $\in \mathbb{Z}$, how do we reconstruct f accurately?

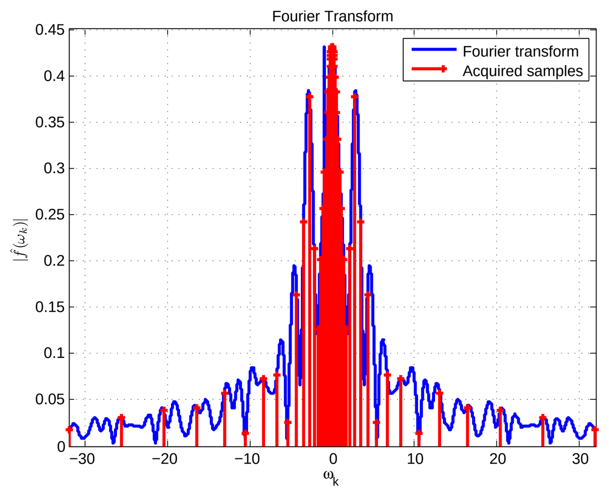

Fig: Sample Acquisition

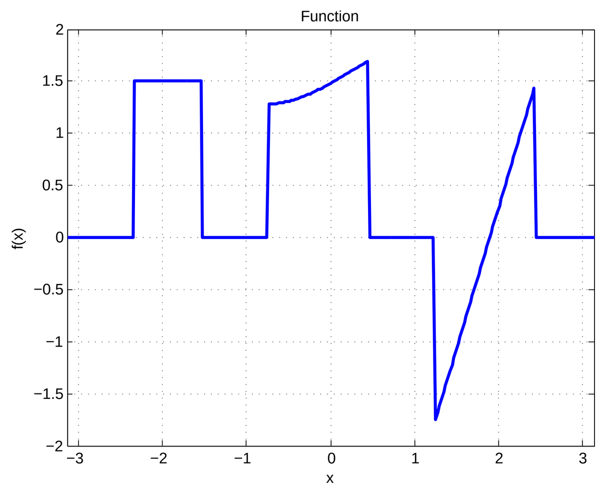

Fig: True Function

- The reconstruction problem is complicated by issues such as small sample size, non-uniform sample density and piecewise-smooth nature of the underlying function.

Non-harmonic Acquisitions

- The problem posed by non-harmonic acquisitions can be understood by looking at the properties of the kernel described by the non-harmonic modes.
- Let $A_N := \sum_{|k|\le N} e^{i\omega_k x}$ be the "non-harmonic" kernel. Examples of this kernel for two different sample acquisitions are plotted below.

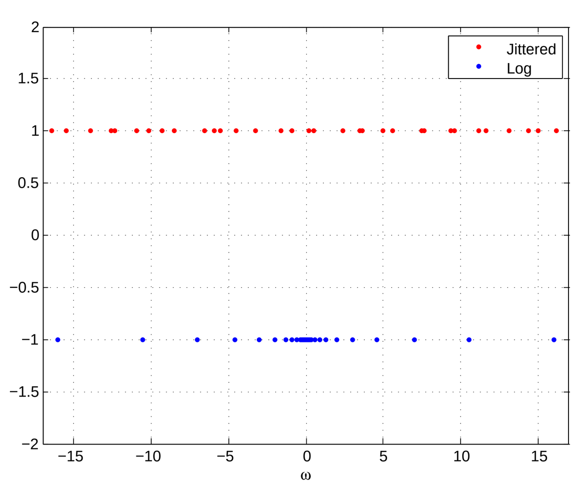

Fig: Sampling Schemes

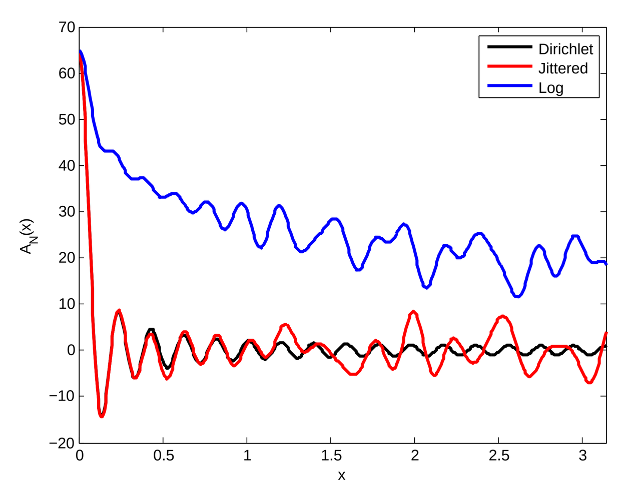

Fig: Kernels

- Depending on the acquired samples, A_N may be non-decreasing and may suffer from poor localization.
- When using conventional Fourier partial sums, these effects appear in the reconstruction, distinct from traditional Gibbs artifacts.

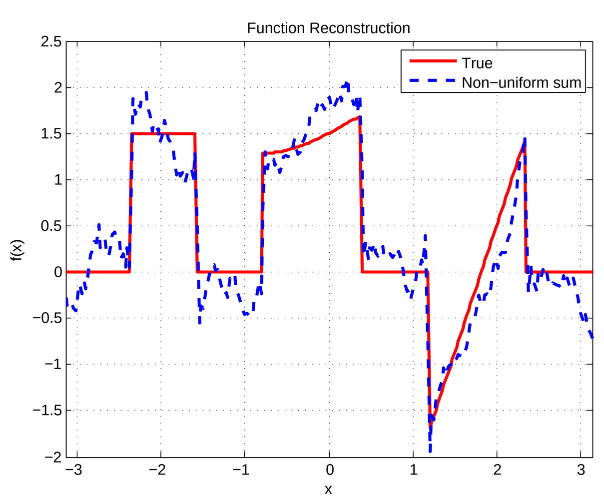

Fig: Reconstruction - Jittered Sampling

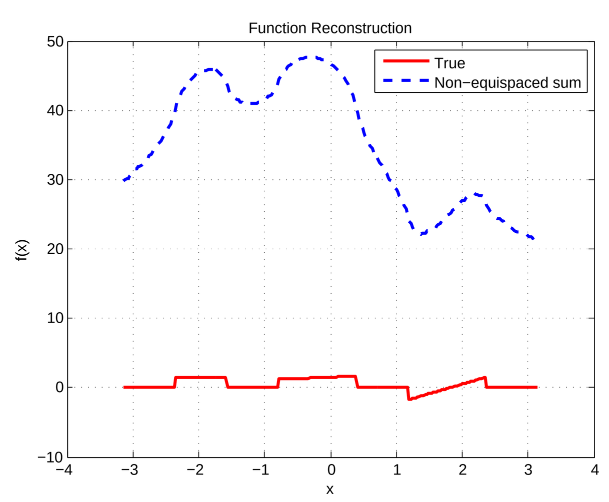

Fig: Reconstruction - Log Sampling

Gridding Reconstruction

- Gridding, [1], maps non-uniform data to a uniform grid using convolution. Let $\hat{f}(\omega_k)$ denote input samples and $\hat{g}(k)$ denote "regridded" coefficients.

$$\hat{g}(k) = (\hat{f} * \hat{\phi})(\omega)\Big|_{\omega=k} \approx \sum_{m \text{ st. } |k-\omega_m|\le q} \alpha_m \hat{f}(\omega_m)\hat{\phi}(k-\omega_m)$$

- ϕ is a window function, q is a neighborhood parameter and α denotes quadrature weights or density compensation factors (DCFs).
- Standard Fourier reconstruction may now be used to recover the physical-space function; i.e.,

$$S_N\tilde{g}(x) = \sum_{k=-N}^{N} \hat{\tilde{g}}(k)e^{ikx}$$

- The function approximation is obtained after dividing out the window.

$$S_N\tilde{f}(x) = \frac{\tilde{g}(x)}{\phi(x)}$$

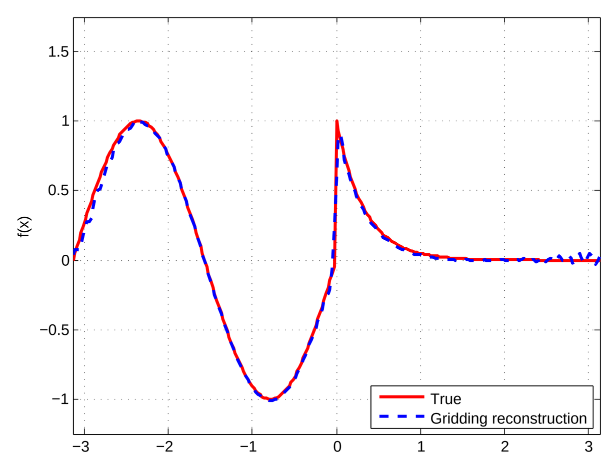

Fig: Example Reconstruction

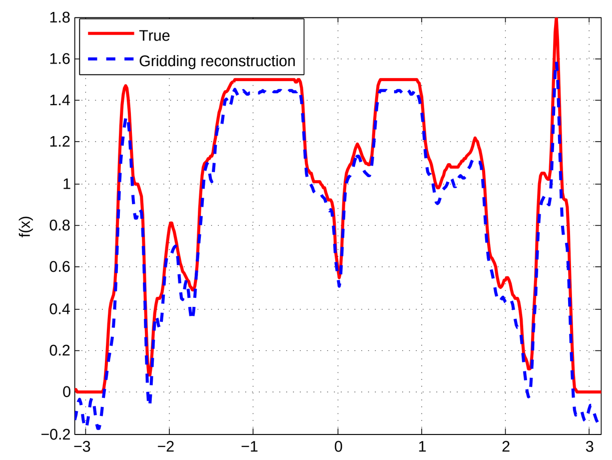

Fig: Reconstruction - Brain Cross-section

- While extensively used, this method typically requires significant filtering to mitigate the effects of non-harmonic acquisition.
- Gridding does not handle non-uniform sampling density well, [2].

Use of Spectral Re-projection Methods

- This framework was originally devised to overcome the Gibbs phenomenon.
- The function is reconstructed in an alternate (fast converging or Gibbs complementary, [3]) basis. The high Fourier modes have exponentially small projections on the low modes of this new basis.
- For Fourier expansions, the Gegenbauer polynomials $\{C_l^\lambda\}$ have been found to form a Gibbs complementary basis.

Reconstruction Procedure

1. Perform convolutional gridding to obtain coefficients on the equispaced grid ($\hat{\tilde{g}}$).
2. Identify smooth regions in g using methods such as the concentration method of edge detection, [4].
3. In each smooth interval:
   1. Re-project Fourier spectral data onto the Gegenbauer basis.

      $$z_l = \langle S_N\tilde{g}, C_l^\lambda\rangle_w = \frac{1}{h_l^\lambda}\int_{-1}^{1}(1-\eta^2)^{\lambda-1/2}C_l^\lambda(\eta)\sum_{|k|\le N}\hat{\tilde{g}}(k)e^{i\pi k\eta}d\eta$$

   2. Function reconstruction using the Gegenbauer partial sum $P_mS_N\tilde{g}$.

      $$P_mS_N\tilde{g}(x) = \sum_{l=0}^{m} z_lC_l^\lambda(x)$$

4. Recover f by dividing out ϕ; i.e., $\tilde{f}(x) = \frac{P_mS_N\tilde{g}(x)}{\phi(x)}$

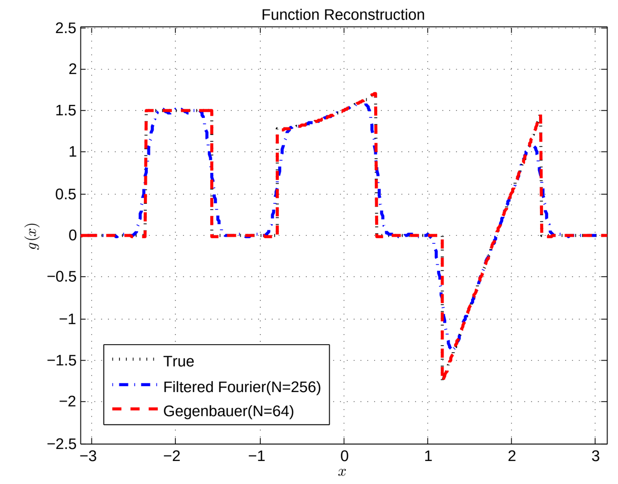

Fig: Reconstruction

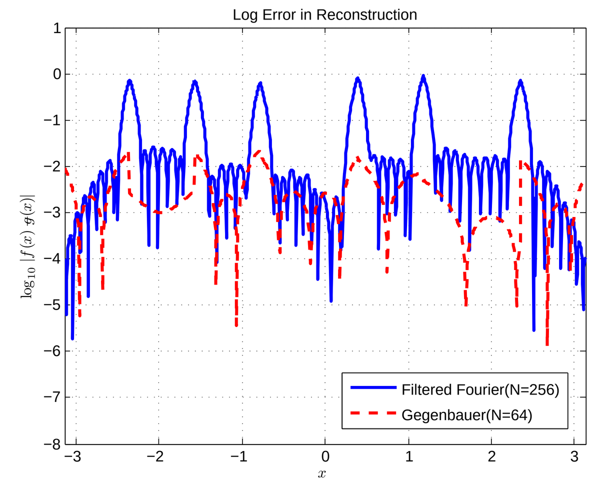

Fig: Log Reconstruction Error

Advantages of Spectral Re-projection

- The re-projection process exponentially damps the error in the high-mode regridded coefficients.
- Gibbs artifacts are also removed, leading to improved reconstruction quality.
- There is evidence to suggest that the spectral re-projection reconstructions require far fewer input measurements.

Current Directions - Using Edge Information

- Integrating the Fourier integral by parts, we obtain

$$\hat{f}(k) = \frac{1}{2\pi}\sum_{a\in\mathcal{A}}\left(\frac{[f](\zeta_a)}{ik} + \frac{[f'](\zeta_a)}{(ik)^2} + \frac{[f''](\zeta_a)}{(ik)^3} + \ldots\right)e^{-ik\zeta_a}$$

  - $[f](\cdot)$ denotes jump values, $[f'](\cdot)$ denotes jump values in the derivative of f, and so on.
  - ζ_a, $a \in \mathcal{A}$ denote the jump locations.
- We note that estimates of these jump locations and values may be obtained by applying the concentration method of edge detection (or one of its variants) on the regridded spectral coefficients.
- Substituting these estimates in the above equation allows us to recover the equispaced Fourier coefficients to within $\mathcal{O}\left(\frac{1}{|k|^2}\right)$ accuracy.
- This significantly improves on the high-mode accuracy of standard gridding.

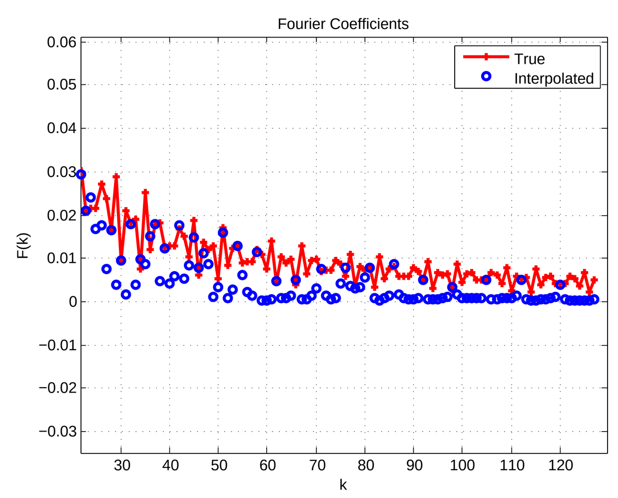

Fig: Standard Gridding

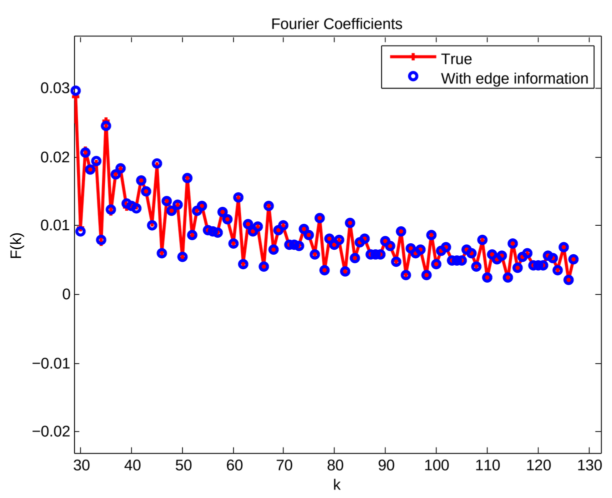

Fig: On Incorporating Edge Information

References and Acknowledgement

1. J. D. O'Sullivan, *Fast Sinc Function Gridding Algorithm for Fourier Inversion in Computer Tomography*, in IEEE Trans. Med. Imag., 4(4):200–207, 1985.
2. A. Viswanathan, A. Gelb, D. Cochran and R. Renaut, *On reconstruction from non-uniform spectral data*, in J. Sci. Comp., (submitted) (Sep. 2009)
3. D. Gottlieb and C. W. Shu, *On the Gibbs Phenomenon and its Resolution*, in SIAM Review, 39(4):644-668, 1997.
4. A. Gelb and E. Tadmor, *Detection of Edges in Spectral Data*, in Appl. Comp. Harmonic Anal., 7: 101–135, 1999.

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