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KADISON–KASTLER STABLE FACTORS

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Abstract. A conjecture of Kadison and Kastler from 1972 asks whether sufficiently close operator algebras in a natural uniform sense must be small unitary perturbations of one another. For $n \geq 3$ and a free ergodic probability measure preserving action of $SL_n(\mathbb{Z})$ on a standard nonatomic probability space (X,μ), write $M = ((L^\infty(X,\mu) \rtimes SL_n(\mathbb{Z}))\,\overline{\otimes}\, R$, where R is the hyperfinite II_1 factor. We show that whenever M is represented as a von Neumann algebra on some Hilbert space $\mathcal{H}$ and $N \subseteq \mathcal{B}(\mathcal{H})$ is sufficiently close to M, then there is a unitary u on $\mathcal{H}$ close to the identity operator with $uMu^* = N$. This provides the first nonamenable class of von Neumann algebras satisfying Kadison and Kastler's conjecture.

We also obtain stability results for crossed products $L^\infty(X,\mu) \rtimes \Gamma$ whenever the comparison map from the bounded to usual group cohomology vanishes in degree 2 for the module $L^2(X,\mu)$. In this case, any von Neumann algebra sufficiently close to such a crossed product is necessarily isomorphic to it. In particular, this result applies when Γ is a free group.

This paper provides a complete account of the results announced in [12].

Contents

1. Introduction

In [51], Kadison and Kastler introduced a metric d on the collection of all closed subalgebras of the bounded operators on a Hilbert space in terms of the Hausdorff distance between the unit balls of two algebras M and N, and conjectured that sufficiently close operator algebras should be isomorphic. Qualitatively, M and N are close in the Kadison-Kastler metric if each operator in the unit ball of M is close to an operator in the unit ball of N and vice versa. Canonical examples of close operator algebras are obtained by small unitary perturbations: given an operator algebra M on a Hilbert space $\mathcal{H}$ and a unitary operator u on $\mathcal{H}$ close to the identity operator, then uMu^* is close to M. The strongest form of the Kadison-Kastler conjecture states that every algebra sufficiently close to a von Neumann

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algebra M arises in this fashion. This has been established when M is an injective von Neumann algebra [18, 89, 44, 21] (building on the earlier special cases in [17, 73]) but remains open for general von Neumann algebras.

We now present the central result of the paper: Theorem A. This has been announced in our short survey article [12] which contains a heuristic discussion of our methods but no formal proofs.

Theorem A. *Let $n \geq 3$ and let $\alpha : SL_n(\mathbb{Z}) \curvearrowright (X,\mu)$ be a free, ergodic and measure preserving action of $SL_n(\mathbb{Z})$ on a standard nonatomic probability space (X,μ). Write $M = (L^\infty(X,\mu) \rtimes_\alpha SL_n(\mathbb{Z}))\,\overline{\otimes}\, R$, where R is the hyperfinite II_1 factor. For $\varepsilon > 0$, there exists $\delta > 0$ with the following property: given a normal unital representation $M \subseteq \mathcal{B}(\mathcal{H})$ and another von Neumann algebra N on $\mathcal{H}$ with $d(M,N) < \delta$, there exists a unitary $u \in \mathcal{H}$ with $\|u - I_{\mathcal{H}}\| < \varepsilon$ and $uMu^* = N$.*

Theorem A provides the first nonamenable II_1 factors which satisfy the strongest form of the Kadison-Kastler conjecture. A key ingredient in this result is the vanishing of the bounded cohomology groups $H^2_b(SL_n(\mathbb{Z}), L^\infty_{\mathbb{R}}(X,\mu))$ for $n \geq 3$ from [61, 9, 63] and in Theorem A, which is then an immediate consequence of Theorem 6.3.4, $SL_n(\mathbb{Z})$ can be replaced with any other group with this property. Via the work of [4, 83, 84, 85], there are uncountably many pairwise nonisomorphic II_1 factors to which this theorem applies (see Remark 6.3.6).

The Kadison-Kastler conjecture is known to be false in full generality. In [16], examples of arbitrarily close nonseparable and nonisomorphic C^*-algebras were found, while in [45] Johnson presented examples of arbitrarily close unitarily conjugate pairs of separable nuclear C^*-algebras where the implementing unitaries could not be chosen to be close to the identity operator. Thus the appropriate form of the conjecture for C^*-algebras is that sufficiently close separable C^*-algebras should be isomorphic or spatially isomorphic. In this last form, the conjecture has been settled affirmatively for close separable nuclear C^*-algebras on separable Hilbert spaces [28] (see also [27]) with earlier special cases established in [21, 74, 75, 55]. Our methods also give examples of nonamenable von Neumann algebras satisfying these weaker forms of the conjecture, as we now state. The hypotheses on the action in the following theorem ensure that M is a II_1 factor with separable predual satisfying $P' \cap M \subseteq P$. The three parts of Theorem B are proved in Section 6 as Corollary 6.1.2, Corollary 6.2.1 and Theorem 6.3.2 respectively.

Theorem B. *Let $\alpha : \Gamma \curvearrowright P$ be a centrally ergodic, properly outer and trace-preserving action of a countable discrete group Γ on a finite amenable von Neumann algebra P with separable predual and write $M = P \rtimes_\alpha \Gamma$.*

(1) Suppose that the comparison map

$$H^2_b(\Gamma, L^2(\mathcal{Z}(P)_{sa})) \to H^2(\Gamma, L^2(\mathcal{Z}(P)_{sa})) \tag{1.1}$$

from bounded cohomology to usual cohomology vanishes, where $\mathcal{Z}(P)$ denotes the center of P. Then, given a normal unital representation $M \subseteq \mathbb{B}(\mathcal{H})$, each von Neumann algebra N on $\mathcal{B}(\mathcal{H})$ sufficiently close to M is isomorphic to M.

(2) Suppose that the comparison map (1.1) vanishes and that M has property Gamma. Then, given a normal unital representation $M \subseteq \mathcal{B}(\mathcal{H})$, each von Neumann algebra N on $\mathcal{B}(\mathcal{H})$ sufficiently close to M is spatially isomorphic to M.

(3) *Suppose that the bounded cohomology group $H^2_b(\Gamma, \mathcal{Z}(P)_{sa})$ vanishes. Then, given $\varepsilon > 0$, there exists $\delta > 0$ such that for a normal unital representation $\iota : M \to \mathcal{B}(\mathcal{H})$ and a von Neumann subalgebra $N \subseteq \mathcal{B}(\mathcal{H})$ with $d(\iota(M), N) < \varepsilon$, there exists a surjective *-isomorphism $\theta : M \to N$ with $\|\iota - \theta\| < \delta$.*

In order to distinguish the slightly different external rigidity properties arising in Theorem A and the different parts of Theorem B above, we call algebras satisfying the conclusion of Theorem A *strongly Kadison-Kastler stable*, algebras satisfying the conclusion of Theorem B part (1) *weakly Kadison-Kastler stable* and algebras satisfying the conclusion of Theorem B part (2) *Kadison-Kastler stable*. With this terminology, the appropriate forms of the Kadison-Kastler conjecture are that von Neumann algebras are strongly Kadison-Kastler stable and separable C^*-algebras are Kadison-Kastler stable.

Part (1) of Theorem B applies when Γ is a free group $\mathbb{F}_r$, $2 \leq r \leq \infty$, as these groups have cohomological dimension one, so $H^2(\Gamma, L^2(\mathcal{Z}(P)_{sa})) = 0$. In particular, the approximate free group factors introduced in [70] as the first class of factors containing a unique Cartan masa up to unitary conjugacy, have the form $L^\infty(X) \rtimes_\alpha \mathbb{F}_r$ for some free ergodic measure preserving profinite action α. Consequently, these factors are weakly Kadison-Kastler stable by Part (1) of Theorem B. As shown in [70, Section 5], there are uncountably many pairwise nonisomorphic factors in this class, including some examples with property Gamma. These latter examples are Kadison Kastler stable by Part 2 of Theorem B.

In the rest of this introduction we set out the main ideas used to prove Theorems A and B for crossed products $P \rtimes_\alpha \Gamma$ arising from centrally ergodic, properly outer and trace-preserving actions α of countable discrete groups Γ on amenable finite von Neumann algebras with separable preduals. Existing methods for isomorphisms between close operator algebras used in [18, 21, 44, 89, 28] when one algebra is amenable rely heavily on perturbation results for approximate homomorphisms. A pair of Banach algebras (A, B) has Johnson's 'AMNM' property [46] when every approximately multiplicative bounded linear map $A \to B$ is near to a multiplicative map. This property is an operator algebra version of Kazhdan's work on almost group representations [54] (see [10] for recent progress in this direction). The AMNM approach to Kadison-Kastler stability results is to produce a bounded linear map between close operator algebras using, for example, a conditional expectation onto an injective von Neumann algebra, and then use the AMNM property to produce a nearby isomorphism. Outside the amenable context it seems difficult to obtain such bounded linear maps and so the approach that we adopt here is to construct directly an isomorphism from a crossed product M onto a close von Neumann algebra N. Our strategy is to establish that algebras N close to a crossed product II_1 factor $M = P \rtimes_\alpha \Gamma$ can be written as twisted crossed products arising from the same underlying action α via a 2-cocycle ω taking values in the unitary group $\mathcal{U}(\mathcal{Z}(P))$. This cocycle ω will be uniformly close to the trivial cocycle, and so one can take a logarithm to obtain a bounded 2-cocycle taking values in $\mathcal{Z}(P)_{sa}$. When this cocycle represents the trivial class in $H^2(\Gamma, \mathcal{Z}(P)_{sa})$, we obtain an isomorphism between M and N. This is guaranteed to occur when the comparison map (1.1) vanishes, or when the bounded cohomology group $H^2_b(\Gamma, \mathcal{Z}(P)_{sa})$ vanishes.

In addition to the perturbation results for amenable von Neumann algebra from [18, 44, 89, 21], two existing perturbation results play a key role in this paper. The first of these (from Kadison and Kastler's original paper [51]) is that close von Neumann algebras have close type decompositions. Of particular relevance here is [51, Theorem B] which shows that if M is a II_1 factor acting nondegenerately on $\mathcal{H}$ and N is another von Neumann algebra acting nondegenerately on $\mathcal{H}$ with $d(M,N) < 1/8$, then N is also a II_1 factor. The second key ingredient is an embedding theorem for near inclusions from [21]. Say that P is nearly contained in N when operators from the unit ball of P can be closely approximated by operators in N but no reverse assumption is made. The analogous conjecture in this situation is that sufficiently small near inclusions of P into N arise from small unitary perturbations of genuine inclusions and the embedding theorem ([21, Theorem 4.3]) establishes this when P is amenable. Further, given such a near inclusion one can find a unitary implementing such a small perturbation inside the von Neumann algebra generated by P and N.

When P is amenable and $M = P \rtimes_\alpha \Gamma$ is close to another von Neumann algebra N on the Hilbert space $\mathcal{H}$, we can apply the embedding theorem to the near containment of P into N. This enables us to replace N by a small unitary perturbation $N_1 = u^*Nu$ of N such that $P \subseteq N_1$. The assumptions on the action α ensure that $P' \cap M \subseteq P$ and it is easily checked that if N_1 is close enough to M, then $P' \cap N_1 \subseteq P$. Writing $(u_g)_{g\in\Gamma}$ for the canonical unitaries in the crossed product $M = P \rtimes_\alpha \Gamma$, we can approximate each u_g by a nearby unitary $\tilde{v}_g \in N_1$. The amenable algebras $\tilde{v}_g P \tilde{v}_g^*$ are each close to P inside N_1, so the perturbation results for amenable von Neumann algebras can be used to replace the unitaries $(\tilde{v}_g)_{g\in\Gamma}$ by unitaries $(v_g)_{g\in\Gamma}$ in N_1 which normalize P and implement the action α. In this way $(P \cup \{v_g : g \in \Gamma\})$ will generate a von Neumann subalgebra of N_1 isomorphic to a twisted crossed product $P \rtimes_{\alpha,\omega} \Gamma$, where ω is a 2-cocycle given by $\omega(g,h) = v_g v_h v_{gh}^*$ for $g, h \in \Gamma$. This is the subject of Section 3.

In order to show that N_1 is generated by P and $(v_g)_{g\in\Gamma}$, we construct special representations of M and N_1 on a new Hilbert space $\mathcal{K}$. Specifically, we reduce to the situation where M and N_1 are close von Neumann algebras on $\mathcal{K}$ with $P \subseteq M \cap N_1$, and both algebras are in standard position. Moreover, we obtain two additional properties. The first is the existence of a common trace vector ξ for M, M', N_1 and N_1' that is then used to define the modular conjugation operators J_M and J_{N_1}. The second is the equality of the two basic construction algebras $\langle M, e_P\rangle$ and $\langle N_1, e_P\rangle$. We are then able to work at the Hilbert space level to convert questions about generation to questions about the orthogonal complement of $\sum_{g\in G} Pv_g$ on $\mathcal{K}$.

To reach this situation we first modify $\mathcal{H}$ to produce a new Hilbert space $\mathcal{K}$ with close representations of M and N_1 such that M is in standard position on $\mathcal{K}$, in the sense that the dimension $\dim_M \mathcal{K}$ of $\mathcal{K}$ as a left M-module is 1. To increase the dimension of $\mathcal{H}$ as an M-module, we can replace $\mathcal{H}$ by an amplification $\mathcal{H} \otimes \mathcal{H}_1$ and M by $M \otimes I_{\mathcal{H}_1}$ and N_1 by $N_1 \otimes I_{\mathcal{H}_1}$. To decrease the dimension of $\mathcal{H}$ as an M-module, we replace $\mathcal{H}$ by $e(\mathcal{H})$ for a suitable projection e in the commutant of M. Provided this projection is chosen near to the commutant of N_1, then we can replace N_1 by a small unitary perturbation of N_1 so that $e \in N_1'$: in this way we obtain close representations of M and N_1 on $e(\mathcal{H})$. The selection of e is a little delicate, as while M and N_1 are close on the original Hilbert space $\mathcal{H}$, we cannot generally assume that the commutants M' and N_1' are also close on $\mathcal{H}$ (see [11]). In Lemma 4.2.1 we show that such an e can nevertheless always be found, though it could be very small in trace.

Once M lies in standard position on $\mathcal{K}$, some ideas originating in [22] and [26] can be used to see that the commutants M' and N_1' are close on $\mathcal{K}$. This enables us to deduce that N_1 is almost in standard position on $\mathcal{K}$ in the sense that, as a left N_1-module, the dimension of $\mathcal{K}$ is approximately 1. Constructing the modular conjugation operator J_M with respect to a trace vector ξ for M and M', it follows that $J_M P J_M$ is almost contained in N_1', so the embedding theorem enables us to make a further small unitary perturbation of N_1 so that $J_M P J_M \subseteq N_1'$. By computing the trace of an operator in N_1 through P via the trace preserving conditional expectation from N_1 onto P and then working in M, we deduce that ξ is also tracial for N_1. Taking commutants and working with the subalgebra $J_M P J_M \subseteq M' \cap N_1'$, it follows that ξ is also tracial for N_1'. In the case that P is a maximal abelian subalgebra of M, it is then easily checked that the basic constructions of $P \subseteq M$ and $P \subseteq N_1$ are identical. In the general case, we use a theorem of Popa [79] to find suitable masas for M and N_1 inside P. At this stage of the argument it is essential that $P' \cap M \subseteq P$. The situation where we assume that M is in standard position is the subject of Section 4.1 and in Section 4.2 we combine this with the reduction procedure of the previous paragraph. In particular it follows that an algebra N sufficiently close to a crossed product $M = P \rtimes_\alpha \Gamma$ can be written as a twisted crossed product $N = P \rtimes_{\alpha,\omega} \Gamma$ for a cocycle ω uniformly close to the trivial cocycle. The weak Kadison-Kastler stability results in part (1) of Theorem B then follow directly; the details are set out in Section 6.1. The procedure of Section 4.2 also gives a general reduction result (Theorem 4.2.4) which shows that, for the purposes of determining whether II_1 factors are weakly Kadison-Kastler stable, it suffices to assume that both factors lie in standard position on the same Hilbert space.

We now discuss how to ensure that the resulting isomorphism is always spatial (as required in Theorem A and part (2) of Theorem B) and uniformly close to the inclusion map (as in Theorem A and part (3) of Theorem B). These problems are intimately connected with Kadison's similarity problem for operator algebras from [49]. We say that a C^*-algebra has the similarity property if every bounded homomorphism $\phi : A \to \mathbb{B}(\mathcal{H})$ is similar to a *-homomorphism. The similarity problem is then to determine whether all C^*-algebras have this property. Positive answers to this question are known for amenable algebras (see [7], for example), traceless C^*-algebras [39], and also in the presence of a cyclic vector [39]. The most general class of II_1 factors for which the similarity property is known to hold are those II_1 factors with Murray and von Neumann's property Gamma [23]; the problem remains open for other II_1 factors. The main result of [26] shows that if $A, B \subseteq \mathcal{B}(\mathcal{H})$ are C^*-algebras, such that A has the similarity property and B is sufficiently close to A (in terms of quantified estimates on how well A satisfies Pisier's length characterization of the similarity property from [76]), then B has the similarity property. From this we obtain that A' and B' are close on $\mathcal{H}$ and that matrices over A are uniformly close to matrices over B (independently of the size of the matrix). In particular, if M and N_1 are close II_1 factors on $\mathcal{H}$ and M has property Gamma, then M' and N_1' are close on $\mathcal{H}$. This enables us to make significant simplifications to the reduction argument of Section 4.2. In particular it follows that $\dim_M(\mathcal{H}) = \dim_{N_1}(\mathcal{H})$ and so if there exists an isomorphism θ between M and N_1, then θ is automatically spatially implemented on $\mathcal{H}$. This is enough to prove part (2) of Theorem B, which is established as Corollary 6.2.1.

In fact, the similarity property for a II_1 factor M is equivalent to the following notion of continuity for the operation of taking commutants: whenever M is normally unitally represented on $\mathcal{H}$ and $N_k \subseteq \mathcal{B}(\mathcal{H})$, $k \geq 1$, is a sequence of von Neumann algebras satisfying $\lim_{k\to\infty} d(M, N_k) = 0$, then $\lim_{k\to\infty} d(M', N_k') = 0$ [11]. In particular, every strongly Kadison-Kastler stable factor must have the similarity property.

In Theorem B the difference between the vanishing of the comparison map (1.1) in part (1) and the hypothesis that $H_b^2(\Gamma, \mathcal{Z}(P)_{sa}) = 0$ in part (3) is that in the latter case we can obtain an isomorphism θ from $M = P \rtimes_\alpha \Gamma$ onto a close subalgebra N with the property that

$$\|\theta(xu_g) - xu_g\| \tag{1.2}$$

is small for $x \in P$ with $\|x\| \leq 1$ and each $g \in \Gamma$ (where $(u_g)_{g\in\Gamma}$ are the canonical group unitaries), whereas when (1.1) vanishes we only learn that the cocycle ω arising in the expression of $N = P \rtimes_{\alpha,\omega} \Gamma$ is a coboundary $\partial\nu$, but we do not have an estimate on how close ν is to the trivial 1-cochain. Consequently, we cannot control the behavior of the isomorphism between M and N on the canonical unitaries in this case.

One would expect that (1.2) is insufficient to ensure that $\|\theta(m) - m\|$ is small for all $m \in M$ with $\|m\| \leq 1$. However, working in the situation where $P \subseteq M \cap N$ and M and N are simultaneously in standard position on a new space $\mathcal{K}$ such that $P \subseteq M$ and $P \subseteq N$ have the same basic construction algebra $\langle M, e_P\rangle$, the isomorphism θ is spatially implemented on $\mathcal{K}$ by a unitary $W \in P' \cap \langle M, e_P\rangle$. By decomposing $W = \sum_{g\in\Gamma} w_g(u_g e_P u_g^*)$ with respect to the central projections $(u_g e_P u_g^*)_{g\in\Gamma}$ in $P' \cap \langle M, e_P\rangle$ arising from the canonical Pimsner-Popa basis, the assumption that $\theta(u_g) \approx u_g$ for all g can be used to show that $\alpha_h(w_{1_\Gamma}) \approx w_h$ for all $h \in \Gamma$. It then follows that $W \approx J_M w_{1_\Gamma} J_M$ and this last unitary lies in $P' \supseteq M' \cap N'$ so W almost commutes with elements of M.

The methods described so far give strong Kadison-Kastler stability for a factor satisfying both the hypotheses of parts (2) and (3) of Theorem B. However, as our principal example $SL_3(\mathbb{Z})$ of a group with the required bounded cohomology condition has property T, it is not possible to produce crossed product factors $L^\infty(X,\mu) \rtimes_\alpha SL_3(\mathbb{Z})$ with property Gamma. Instead we obtain our examples of strongly Kadison-Kastler stable factors by taking a tensor product of $L^\infty(X,\mu) \rtimes_\alpha SL_3(\mathbb{Z})$ and the hyperfinite II_1 factor R in order to obtain property Gamma. In section 5.2 we show that algebras close to McDuff factors (those of the form $M = M_0 \overline{\otimes} R$) are again of this form, and further, we can arrange to factor an algebra N close to $M_0 \overline{\otimes} R$ as $N_0 \overline{\otimes} R$ using the same copy of R and with N_0 close to M_0. This enables the work described previously to be applied to M_0 and N_0 and prove Theorem A as Theorem 6.3.4. This is set out in Section 6.3.

The method of identifying N as a twisted crossed product coming from the same underlying action used to construct M, albeit with a possibly different cocycle, is also valid beyond the crossed product setting. In [38], Feldman and Moore introduced the notion of a Cartan masa in a von Neumann algebra to generalize the twisted version of Murray and von Neumann's group-measure space construction and showed that these are determined by a measured equivalence relation and the orbit of a certain 2-cocycle. Our results enable us to see that if $A \subseteq M$ is an inclusion of a Cartan masa in a II_1 factor, then any von Neumann algebra N sufficiently close to M contains a Cartan masa B close to A. Further, the inclusions $A \subseteq M$ and $B \subseteq N$ induce isomorphic measured equivalence relations and are given by uniformly close 2-cocycles on these relations (see Subsection 6.4 and Proposition 6.4.2). In particular, factors close to those with unique Cartan masas (see [70, 71, 14, 87, 88]) also have unique Cartan masas. Our methods then give isomorphism results for close II_1 factors $M = L^\infty(X,\mu) \rtimes \Gamma$ and $N = L^\infty(Y,\nu) \rtimes \Lambda$ which both arise as crossed products in a class which is known to have a unique Cartan masa. In particular, Popa and Vaes have shown that this applies when Γ and Λ are both hyperbolic [88] (see [14] when the actions are additionally assumed profinite), whereas general hyperbolic groups need not satisfy the cohomological condition that the comparison map (1.1) vanishes.

The paper is organized as follows. In order to make this paper as self-contained as possible, we start with a background section which reviews the Kadison-Kastler metric, near inclusions and the connection with the similarity problem, the basic construction, the twisted crossed product construction and the bounded cohomology results which apply to $SL_n(\mathbb{Z})$ for $n \geq 3$. At this point we recall some prior estimates from the literature and establish some technical lemmas. We also collect a number of easy estimates and some technical results for later use. Section 3 is concerned with the process of transferring normalizers of amenable subalgebras. We do this in a general setting designed to work beyond the context of crossed products. Section 4 contains the procedure for representing two close II_1 factors on a new Hilbert space where both are in standard position, and other important properties of this reduction are also obtained. Section 5 investigates structural properties of close II_1 factors in the sprit of the original paper [51]. Using the work of Sections 3 and 4, we show that a number of key structural properties of the free group factors, including strong solidity, are inherited by nearby factors, and then examine factors with property Gamma and McDuff factors. Theorems A and B are established in Section 6 which also sets out how our work applies to general Cartan masas through the lens of equivalence relations.

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2. Preliminaries

2.1. **The Kadison-Kastler metric and near inclusions.**

We begin by recalling the definition of the Kadison-Kastler metric from [51] and by using it to give precise formulations of the concept of Kadison-Kastler stability from the introduction. We state these definitions in the context of von Neumann algebras, but they apply equally to C^*-algebras (replacing the unital assumption by a nondegeneracy assumption), and to non self-adjoint algebras (where spatial isomorphism should be interpreted in terms of conjugation by a similarity rather than by a unitary).

Definition 2.1.1. Let M and N be von Neumann subalgebras of $\mathcal{B}(\mathcal{H})$. The *Kadison-Kastler distance* $d(M,N)$ between M and N is the infimum of those $\gamma > 0$ with the property that, given an operator x in one of the unit balls of M or N, there exists y in the other unit ball with $\|x-y\| < \gamma$.

Given two von Neumann algebras M and N acting degenerately on $\mathcal{H}$ with $d(M,N)$ small, it is easy to show that the identities I_M and I_N are close in norm. Further, there exists a unitary $u \in W^*(M \cup N)$ with $uI_Mu^* = I_N$ so that $\|u - I_{\mathcal{H}}\|$ is small (see [26, Proposition 3.2], which sets this out with explicit estimates in the more general C^*-algebraic context). We can then replace N by the algebra $N_1 = u^*Nu$ and work on the Hilbert space $\mathcal{H}_1 = I_M(\mathcal{H})$, where M and N_1 act nondegenerately. Thus we can assume that close von Neumann algebras share the same unit and that this is the identity operator on the underlying Hilbert space. We incorporate this assumption into the definitions below.

Definition 2.1.2. Let M be a von Neumann algebra.

(i) Say that M is *strongly Kadison-Kastler stable* if for all $\varepsilon > 0$, there exists $\delta > 0$ such that given any faithful unital normal representation $M \subseteq \mathcal{B}(\mathcal{H})$ and a von Neumann algebra $N \subseteq \mathcal{B}(\mathcal{H})$ containing $I_{\mathcal{H}}$ with $d(M,N) < \delta$, then there exists a unitary operator u on $\mathcal{H}$ with $uMu^* = N$ and $\|u - I_{\mathcal{H}}\| < \varepsilon$.

(ii) Say that M is *Kadison-Kastler stable* if there exists $\delta > 0$ such that given a faithful unital normal representation $M \subseteq \mathbb{B}(\mathcal{H})$ and a von Neumann algebra $N \subseteq \mathcal{B}(\mathcal{H})$ with $I_{\mathcal{H}} \in N$ such that $d(M,N) < \delta$, then there exists a unitary operator u on $\mathcal{H}$ with $uMu^* = N$.

(iii) Say that M is *weakly Kadison-Kastler stable* if there exists $\delta > 0$ such that given a faithful unital normal representation $M \subseteq \mathcal{B}(\mathcal{H})$ and a von Neumann algebra $N \subseteq \mathcal{B}(\mathcal{H})$ with $I_{\mathcal{H}} \in N$ such that $d(M,N) < \delta$, then M and N are *-isomorphic.

We now turn to the concept of near inclusions from [21].

Definition 2.1.3. Let $M, N \subseteq \mathcal{B}(\mathcal{H})$ be von Neumann algebras. For $\gamma > 0$, write $M \subseteq_\gamma N$ if each $x \in M$ can be approximated by some $y \in N$ with $\|x - y\| \leq \gamma\|x\|$. Write $M \subset_\gamma N$ when there exists $\gamma' < \gamma$ with $M \subseteq_{\gamma'} N$.

Since the definition of a near inclusion above does not require that the approximating y satisfies $\|y\| \leq \|x\|$, the near inclusions $M \subset_\gamma N$ and $N \subset_\gamma M$ only imply that $d(M,N) < 2\gamma$. Similarly

$$M \subset_\gamma N \subset_\delta P \implies M \subset_{\gamma+\delta+\delta\gamma} P, \tag{2.1}$$

and so, while the infimum of those γ with $M \subset_\gamma N$ and $N \subset_\gamma M$ defines a notion of distance between operator algebras equivalent to the Kadison and Kastler metric d, such a notion does not appear to satisfy the triangle inequality. However, when one of the near inclusions is obtained by conjugation by a unitary, we have a better estimate which will be used repeatedly in the sequel:

$$M \subseteq_\gamma N, \quad u \in \mathcal{U}(\mathbb{B}(\mathcal{H})) \implies M \subseteq_{\gamma+2\|u-I\|} uNu^*. \tag{2.2}$$

This is obtained as follows. For x in the unit ball of M, choose $y \in N$ with $\|x - y\| \leq \gamma$. Then $\|x - uyu^*\| \leq \|x - uxu^*\| + \|u(x - y)u^*\| \leq 2\|u - I\| + \gamma$.

In [21] it is observed that near inclusions behave better than the metric d with respect to operations like matrix amplifications and taking commutants (see Proposition 2.1.6 below). To avoid the repeated losses in converting from near inclusions to distance estimates, we shall state technical results using hypotheses of the form $M \subset_\gamma N$ and $N \subset_\gamma M$, but formulate the main results of the paper using the Kadison-Kastler metric d.

In Section 3, we will need to be able to take commutants of a sufficiently small near inclusion $M \subseteq_\gamma N$ to obtain a reverse near inclusion $N' \subseteq_{\gamma'} M'$. This can be done with constants independent of the underlying Hilbert space $\mathcal{H}$ if and only if M has the similarity property [11]. Without the similarity property, this can be achieved on a given Hilbert space $\mathcal{H}$ when M has a finite set of cyclic vectors for $\mathcal{H}$, with constants depending on the size of a cyclic set [22]. This proceeds by using the noncommutative Grothendieck inequality to show that, in the presence of a cyclic vector, bounded derivations are automatically completely bounded. We give an alternative proof of this fact below, which refines these ideas to give improved constants (our main results use these estimates repeatedly so this leads to a significant improvement in our final bounds). We start by isolating a technical lemma.

Lemma 2.1.4. *Let $M \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra such that $\mathcal{H}$ has a cyclic vector for M. Then, for every derivation $\delta : M \to \mathbb{B}(\mathcal{H})$, we have*

$$\|\delta\|_{cb} \leq 2\|\delta\|_{\mathrm{row}}, \tag{2.3}$$

where $\|\delta\|_{\mathrm{row}}$ denotes the row norm *of δ, given by $\|\delta\|_{\mathrm{row}} = \sup_{n,r} \|\delta_{1\times n}(r)\|$, where the supremum is taken over contractions $r \in \mathbb{M}_{1\times n}(M)$.*

Proof. Let $n \in \mathbb{N}$ and $x \in \mathbb{M}_n(M)$. Since M has a cyclic vector for $\mathcal{B}(\mathcal{H})$, we have

$$\|\delta_n(x)\| = \sup\{\|r\delta_n(x)\| : r \in \mathbb{M}_{1\times n}(M),\ \|r\| \leq 1\}, \tag{2.4}$$

by [78, Lemma 2.4 (i) $\implies$ (iv) and Theorem 2.7]. For a $1 \times n$ row matrix r over M with $\|r\| \leq 1$, the relation $r\delta_n(x) = \delta_{1\times n}(rx) - \delta_{1\times n}(r)x$ leads to the estimate

$$\|r\delta_n(x)\| = \|\delta_{1\times n}(rx) - \delta_{1\times n}(r)x\| \leq 2\|\delta\|_{\mathrm{row}}\|x\|. \tag{2.5}$$

Combining (2.4) and (2.5) gives $\|\delta_n(x)\| \leq 2\|\delta\|_{\mathrm{row}}$, from which the result follows. $\square$

An operator $T \in \mathcal{B}(\mathcal{H})$ induces a derivation $S \mapsto TS - ST$ on $\mathbb{B}(\mathcal{H})$ denoted $\mathrm{ad}(T)$. Arveson's distance formula from [3] (see [19, Proposition 2.1] for the formulation we use) shows that for a von Neumann algebra $M \subseteq \mathcal{B}(H)$ and $T \in \mathbb{B}(H)$,

$$d(T, M') = \frac{1}{2}\|\mathrm{ad}(T)|_M\|_{cb}. \tag{2.6}$$

In the proof below we use [94, Proposition 4.2]. We take this opportunity to correct an oversight in the statement of this result, which omitted the hypothesis that M is finite (which is required in order to appeal to [25, Theorem 2.3] in the proof of [94, Proposition 4.2]). It is for this reason we have to handle the finite and properly infinite cases separately below. It would be interesting to find the best constant in (2.7).

Lemma 2.1.5. *Let $M \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra with a cyclic set of m vectors (in the sense that there exist $\xi_1, \ldots, \xi_m \in \mathcal{H}$ such that* $\mathrm{Span}\{x_i\xi_i : x_i \in M\ \ 1 \leq i \leq m\}$ *is dense in $\mathcal{H}$). If $\delta : M \to \mathbb{B}(\mathcal{H})$ is a bounded derivation, then δ is completely bounded and*

$$\|\delta\|_{cb} \leq 2(1 + \sqrt{2})m\|\delta\|. \tag{2.7}$$

Proof. We first look at the case when $m = 1$ and M has a separable predual. By [22, Corollary 3.2], and Arveson's distance formula (2.6) it suffices to prove the result for when δ is of the form $\mathrm{ad}(T)|_M$ for a fixed $T \in \mathcal{B}(\mathcal{H})$.

Let p be the central projection in M such that $M_0 = Mp$ is finite and $M_1 = M(1-p)$ is properly infinite. Since M_0 has separable predual, results of Popa from [79] can be used to choose an amenable von Neumann algebra $P_0 \subseteq M_0$ with $P_0' \cap M_0 = \mathcal{Z}(M_0)$ (when M_0 is a II_1 factor this is [79, Corollary 4.1], the extension to the case when M_0 is II_1 can be found in [95, Theorem 8], and the general case is obtained by splitting M_0 as a direct sum of its II_1 part and its finite type I part, which is already amenable). Let P_1 be a properly infinite amenable von Neumann subalgebra of M_1 and let $P = (P_0 \cup P_1)''$, which is an amenable von Neumann subalgebra of $\mathcal{B}(\mathcal{H})$ containing p.

Since P is amenable, we can find S in the weak* closed convex hull $\overline{\mathrm{co}}^w\{uTu^* : u \in \mathcal{U}(P)\}$ such that $S \in P'$. By construction (see [19, Section 2]) $\|\mathrm{ad}(S)|_M\| \leq \|\mathrm{ad}(T)|_M\|$ and

$\|S-T\| \le \|\mathrm{ad}(T)|_M\|$. Further, just as in [19, Theorem 2.4], $\{S(1-p),(1-p)S^*\}' \cap M_1$ is properly infinite so

$$\|\mathrm{ad}(S(1-p))|_{M_1}\| = 2d(S(1-p), M_1') = \|\mathrm{ad}(S(1-p))|_{M_1}\|_{cb} \tag{2.8}$$

by [19, Corollary 2.2]. The map $\mathrm{ad}(Sp)|_{M_0}$ is a P_0-module map $M_0 \to \mathcal{B}(p(\mathcal{H}))$ and so by [94, Theorem 4.2], $\mathrm{ad}(Sp)|_{M_0}$ is row bounded with row norm

$$\|\mathrm{ad}(Sp)|_{M_0}\|_{\mathrm{row}} \le \sqrt{2}\|\mathrm{ad}(Sp)|_{M_0}\|. \tag{2.9}$$

Combining (2.8) and (2.9), $\mathrm{ad}(S)|_M$ is row bounded and

$$\|\mathrm{ad}(S)|_M\|_{\mathrm{row}} \le \sqrt{2}\|\mathrm{ad}(S)|_M\|. \tag{2.10}$$

By Lemma 2.1.4, $\mathrm{ad}(S)|_M$ is completely bounded and

$$\|\mathrm{ad}(S)|_M\|_{cb} \le 2\sqrt{2}\|\mathrm{ad}(S)|_M\| \le 2\sqrt{2}\|\mathrm{ad}(T)|_M\|. \tag{2.11}$$

Since $\|T-S\| \le \|\mathrm{ad}(T)|_M\|$, we have

$$\begin{aligned} \|\mathrm{ad}(T)|_M\|_{cb} &\le \|\mathrm{ad}(S)|_M\|_{cb} + \|\mathrm{ad}(T-S)|_M\|_{cb} \\ &\le 2\sqrt{2}\|\mathrm{ad}(T)|_M\| + 2\|T-S\| \le 2(1+\sqrt{2})\|\mathrm{ad}(T)|_M\|, \end{aligned} \tag{2.12}$$

proving the result in the case when $m=1$ and M has separable predual.

We now retain the assumption that M has a cyclic vector ξ but no longer require M to have separable predual. Given a bounded derivation $\delta : M \to \mathcal{B}(\mathcal{H})$, we argue by contradiction, so suppose that $\|\delta\| = 1$ but there is an $n \times n$ matrix $x_0 \in \mathbb{M}_n \otimes M$ of norm 1 with entries from M so that $\|\delta_n(x_0)\| > 2(1+\sqrt{2})$. Let A_0 be the separable C^*-subalgebra of M generated by the entries of x_0 and the identity operator. There exists a finite dimensional subspace $\mathcal{H}_0$ of $\mathcal{H}$ such that, if p_0 is the orthogonal projection from $\mathcal{H}$ onto $\mathcal{H}_0$, then $\|(I_n \otimes p_0)\delta_n(x_0)(I_n \otimes p_0)\| > 2(1+\sqrt{2})$. Since ξ is a cyclic vector for M on $\mathcal{H}$, we can find a separable C^*-subalgebra A_1 of M with $A_0 \subseteq A_1$ such that $\overline{A_1\xi} \supseteq \mathcal{H}_0$. Write p_1 for the orthogonal projection from $\mathcal{H}$ onto $\overline{A_1\xi}$. Now inductively construct a sequence $A_1 \subseteq A_2 \subseteq \dots$ of separable C^*-algebras and projections $p_n \in A_n'$ from $\mathcal{H}$ onto $\overline{A_n\xi}$ so that $x \mapsto xp_n$ is faithful on A_{n-1}. Let A be the direct limit of this sequence, and $p \in A'$ be the projection onto $\overline{A\xi}$. Then A is separable, and $x \mapsto xp$ is faithful on A (as this map is isometric on each A_n).

We can define a derivation $\hat{\delta} : Ap \to \mathcal{B}(\mathcal{K})$ by $\hat{\delta}(ap) = p\delta(a)p$. This is well defined and contractive (as $x \mapsto xp$ is faithful on A) and $\hat{\delta}$ extends with the same norm to a derivation, also denoted $\hat{\delta}$, defined on the ultraweak closure N of Ap by [50, Lemma 3], see also [93, Theorem 2.2.2] (this result is stated for derivations taking values in Ap, but the proofs given in these references work for derivations taking values in $\mathcal{B}(\mathcal{K})$). By construction, N has separable predual (since $\mathcal{K}$ is separable) and a cyclic vector ξ. From the first part of the proof we conclude that $\|\hat{\delta}\|_{cb} \le 2(1+\sqrt{2})$. In particular, $\|(I_n \otimes p)\delta_n(x_0)(I_n \otimes p)\| = \|\hat{\delta}_n(x_0)\| \le 2(1+\sqrt{2})$, giving the required contradiction.

The situation where M has a cyclic set of m vectors can be reduced to the case of a single cyclic vector by tensoring by $\mathbb{M}_m$, see [26, Proposition 2.6]. □

In the context of near inclusions, the previous lemma has the following form.

Proposition 2.1.6. *Let M be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$ and suppose that M has a finite cyclic set of m vectors for $\mathcal{H}$. Then:*

(i) *For $T \in \mathcal{B}(\mathcal{H})$, we have*

$$\inf\{\|T-S\| : S \in M'\} \leq (1+\sqrt{2})m\|\mathrm{ad}(T)|_M\|. \tag{2.13}$$

(ii) *Given another von Neumann algebra N on $\mathcal{H}$ with $M \subseteq_\gamma N$, we have $N' \subseteq_{2(1+\sqrt{2})m\gamma} M'$.*

Note that by taking a weak*-limit point, the infimum on the left hand side in (2.13) is attained.

Proof. Part (i) follows from Lemma 2.1.5 and Arveson's distance formula in (2.6) above. For (ii), note that if N is another von Neumann algebra on $\mathcal{H}$ with $M \subseteq_\gamma N$, then for $T \in N'$ the near inclusion $M \subset_\gamma N$ gives the estimate $\|\mathrm{ad}(T)|_M\| \leq 2\|T\|\gamma$ (see the proof of [21, Theorem 3.1]). The result now follows from (i). □

2.2. **The completely bounded Kadison-Kastler metric.** A completely bounded version of the Kadison-Kastler metric has existed implicitly since [21], and explicitly appears in [26, 90]. As the basic properties of this metric have not been set out we do this below.

Definition 2.2.1. Let $A, B \subseteq \mathcal{B}(\mathcal{H})$ be operator algebras. Write

$$d_{cb}(A,B) = \sup_{n\geq 1} d(A \otimes \mathbb{M}_n, B \otimes \mathbb{M}_n). \tag{2.14}$$

For $\gamma > 0$, write $A \subseteq_{cb,\gamma} B$ if $A \otimes \mathbb{M}_n \subseteq_\gamma B \otimes \mathbb{M}_n$ for all $n \in \mathbb{N}$ and $A \subset_{cb,\gamma} B$ if there exists $\gamma' < \gamma$ with $A \subseteq_{cb,\gamma'} B$.

The following properties follow quickly from these definitions and existing theory.

Properties 2.2.2. (i) *The inequalities*

$$\begin{aligned} &\inf\{\gamma > 0 : A \subset_{cb,\gamma} B,\ B \subset_{cb,\gamma} A\} \\ \leq &d_{cb}(A,B) \leq 2\inf\{\gamma > 0 : A \subset_{cb,\gamma} B,\ B \subset_{cb,\gamma} A\} \end{aligned} \tag{2.15}$$

follow, just as for the original metric.

(ii) *If A and B are C^*-algebras, with $A \subset_{cb,\gamma} B$, then for any nuclear C^*-algebra D, an argument using the completely positive approximation property as in* [21, Theorem 3.1] *yields $A \otimes D \subset_{cb,\gamma} B \otimes D$.*

(iii) *Let $M, N \subseteq \mathcal{B}(\mathcal{H})$ be von Neumann algebras with $M \subset_{cb,\gamma} N$, and let S be any amenable von Neumann algebra. Applying (ii) to any weak* dense nuclear C^*-algebra $D \subseteq S$ and using the Kaplansky density argument of* [51, Lemma 5] *gives $M \overline{\otimes} S \subset_{cb,\gamma} N \overline{\otimes} S$. In particular $M \overline{\otimes} \mathbb{B}(\mathcal{K}) \subset_{cb,\gamma} N \overline{\otimes} \mathcal{B}(\mathcal{K})$.*

Arveson's distance formula [3] shows that completely close algebras have completely close commutants.

Proposition 2.2.3. *Let A and B be C^*-algebras acting nondegenerately on a Hilbert space $\mathcal{H}$ with $A \subseteq_{cb,\gamma} B$. Then $B' \subseteq_{cb,\gamma} A'$. In particular, if $d_{cb}(A,B) \leq \gamma$, then $d_{cb}(A',B') \leq 2\gamma$.*

Proof. Given $n \in \mathbb{N}$ and $T \in \mathcal{B}(\mathcal{H} \otimes \mathbb{C}^n)$, recall that the statement of Arveson's distance formula in (2.6) shows that the distance from T to $(A \otimes \mathbb{C}I_n)' = A' \otimes \mathbb{M}_n$ is given by

$$d(T, A' \otimes \mathbb{M}_n) = \frac{1}{2}\|\mathrm{ad}(T)|_{A\otimes \mathbb{C}I_n}\|_{cb}. \tag{2.16}$$

Now suppose that $T \in B' \otimes \mathbb{M}_n$. Given $x \in A \otimes \mathbb{M}_n \otimes \mathbb{M}_s \cong A \otimes \mathbb{M}_{ns}$, choose $y \in B \otimes \mathbb{M}_{ns}$ with $\|x - y\| \leq \gamma\|x\|$. Then

$$\begin{aligned} \|(\mathrm{ad}(T)|_{A\otimes\mathbb{M}_n} \otimes \mathrm{id}_{\mathbb{M}_s})(x)\| &= \|\mathrm{ad}(T \otimes I_s)(x)\| \\ &\leq 2\|T \otimes I_s\|\|x - y\| \leq 2\|T\|\gamma\|x\|. \end{aligned} \tag{2.17}$$

In particular, this holds for $x \in A \otimes \mathbb{C}I_n \otimes \mathbb{M}_s$, and so (2.16) gives $d(T, A' \otimes \mathbb{M}_n) \leq \|T\|\gamma$, as claimed.

The second statement is an immediate consequence of the previous paragraph and Properties 2.2.2 (i). $\square$

A von Neumann algebra M is said to have property D_k^* for some $k > 0$ if, for every faithful normal unital representation $\pi : M \to \mathcal{B}(\mathcal{H})$ we have $d(T, \pi(M)') \leq k\|\mathrm{ad}(T)|_{\pi(M)}\|$ for $T \in B(\mathcal{H})$. When M is a II_1 factor, the existence of such a k is equivalent to the similarity property (this combines [57], and the folklore result that it suffices to consider normal derivations and representations in the derivation problem and similarity problem for II_1 factors respectively). Further, it dates back to [21] that property D_k^* enables one to take commutants of near inclusions of von Neumann algebras. Here we note that this argument automatically gives completely bounded near inclusions of commutants and record the estimates in the context of property Gamma factors for later use.

Proposition 2.2.4. *Let $M, N \subseteq \mathcal{B}(\mathcal{H})$ be von Neumann algebras acting nondegenerately on $\mathcal{H}$.*

(i) Suppose that M has property D_k^* for some $k > 0$ and $M \subseteq_\gamma N$ for some $\gamma > 0$. Then $N' \subseteq_{cb,2k\gamma} M'$ and $M \subseteq_{cb,2k\gamma} N$.

(ii) Suppose that M is a II_1 factor with property Gamma and $M \subseteq_\gamma N$ for some $\gamma > 0$. Then $N' \subseteq_{cb,5\gamma} M'$ and $M \subseteq_{cb,5\gamma} N$.

Proof. (i). Fix $n \in \mathbb{N}$ and $T \in N' \otimes \mathbb{M}_n \subseteq \mathbb{B}(\mathcal{H} \otimes \mathbb{C}^n)$. Applying property D_k^* to the amplification $M \otimes I_n$ acting on $\mathcal{H} \otimes \mathbb{C}^n$, we see that $d(T, M' \otimes \mathbb{M}_n) \leq k\|\mathrm{ad}(T)|_{M\otimes I_n}\| \leq 2k\gamma\|T\|$ (where the last estimate arises from $M \subseteq_\gamma N$ just as in [21, Theorem 3.1]). Thus $N' \subseteq_{cb,2k\gamma} M'$ and so $M \subseteq_{cb,2k\gamma} N$ by Proposition 2.2.3.

(ii). This now follows immediately as II_1 factors with property Gamma have length at most 5 and length constant (at length 5) 1 by [77, Theorem 13] and hence property $D_{5/2}^*$ (see [76, Remark 4.7] and [26, Section 2] which show that these conditions imply the stronger property $D_{5/2}$ in which we do not restrict to normal representations). $\square$

Remark 2.2.5. The similarity degree, and hence length, of a II_1 factor with property Gamma is 3 ([24]), but at present we do not know how to use this result to improve the estimates in Proposition 2.2.4.

The similarity property, in its derivation form, can also be used to show that small isomorphisms are necessarily spatially implemented. The first part of the lemma below is obtained by making minor changes to the proof of [19, Proposition 3.2] (which handles the case of properly infinite von Neumann algebras using property $D_{3/2}^*$, and McDuff factors using property $D_{5/2}^*$). As property Gamma factors have property $D_{5/2}^*$, the second statement is an immediate consequence of the first.

Lemma 2.2.6. [19] *Let M be von Neumann algebra acting nondengenerately on $\mathcal{H}$ and suppose that $\theta : M \to \mathcal{B}(\mathcal{H})$ is a *-homomorphism with $\|\theta(x) - x\| \leq \gamma\|x\|$ for $x \in M$.*

(i) *Suppose that M has property D_k^* for some $k \geq 1$ and that $\gamma < 1/k$. Then there exists a unitary u on $\mathcal{H}$ such that $\theta = \mathrm{Ad}(u)$ and*

$$\|I_{\mathcal{H}} - u\| \leq 2^{1/2}k\gamma\left(1 + (1-(k\gamma)^2)^{1/2}\right)^{-1/2} \leq 2^{1/2}k\gamma. \tag{2.18}$$

(ii) *Suppose that M has property Gamma and that $\gamma < 2/5$. Then there exists a unitary u on $\mathcal{H}$ such that $\theta = \mathrm{Ad}(u)$ and $\|I_{\mathcal{H}} - u\| \leq 2^{-1/2}5\gamma$.*

2.3. **Standard position, the basic construction and coupling constants.** Let M be a II_1 factor on a Hilbert space $\mathcal{H}$, and let τ (or τ_M when it is necessary to indicate the underlying factor M) denote the unique normal normalized trace on M. If M has a finite commutant on $\mathcal{H}$ and there is a vector $\xi \in \mathcal{H}$ which is tracial for M and M' in the sense that the vector state $\langle \cdot\xi, \xi\rangle$ gives the traces on both M and M', then M is said to be in standard position. Having specified a tracial vector ξ for M in standard position, there is a conjugate linear isometry J (or J_M when we need to emphasize its relation to M) defined by extending $J(m\xi) = m^*\xi$ to $\mathcal{H}$. This is called the conjugation operator, and has the property that $JMJ = M'$. Note that if M is in standard position on $\mathcal{H}$, then any trace vector ξ for M is automatically tracial for M'.

If M is in standard position on $\mathcal{H}$ with a specified tracial vector ξ and $P \subseteq M$ is any von Neumann subalgebra, then e_P denotes the projection onto $\overline{P\xi}$. The basic construction is then defined to be $(M \cup \{e_P\})''$ and is denoted by $\langle M, e_P\rangle$. This algebra dates back to [98] and [20], but was first used systematically in the work of Jones [47]. For later use, we list a few standard properties of the basic construction below which can be found in [47] or standard references (such as [48]) and then record a technical lemma.

Properties 2.3.1. *With the notation above:*

(i) *$e_P = Je_PJ$;*
(ii) *e_P commutes with P and $P = M \cap \{e_P\}'$;*
(iii) *$\langle M, e_P\rangle' = J_MPJ_M$;*
(iv) *The map $p \mapsto pe_P$, $p \in P$, is an algebraic isomorphism, and consequently isometric;*
(v) *For each $x \in M$, $e_Pxe_P = E_P^M(x)e_P$, where E_P^M denotes the unique trace preserving conditional expectation from M onto P.*

Lemma 2.3.2. *Let M be a II_1 factor with separable predual, in standard position on a Hilbert space $\mathcal{H}$ with tracial vector ξ. Let P be a von Neumann subalgebra of M satisfying $P' \cap M \subseteq P$. Given a unitary $v \in P' \cap \langle M, e_P\rangle$, there exists a unique unitary $u \in P' \cap M = \mathcal{Z}(P)$ such that $v\xi = u\xi$.*

Proof. By [35, Lemma 3.2], e_P is central in $P' \cap \langle M, e_P\rangle$, and so $ve_P = e_Pve_P$. Now $e_Pve_P \in Pe_P$ and so there exists an element $u \in P$ such that $e_Pve_P = ue_P$. Since $\|e_Pve_P\| \leq 1$, it follows that $\|u\| \leq 1$. Additionally,

$$v\xi = ve_P\xi = e_Pve_P\xi = ue_P\xi = u\xi, \tag{2.19}$$

and so $\|u\xi\|_{\mathcal{H}} = \|v\xi\|_{\mathcal{H}} = 1$. Combining this with $\|u\| \leq 1$, we see that u is a unitary since $I_{\mathcal{H}} - u^*u \geq 0$ and has zero trace. If $y \in P$ is arbitrary, then

$$uy\xi = uye_P\xi = ue_Py\xi = e_Pve_Py\xi = ye_Pve_P\xi = yue_P\xi = yu\xi, \tag{2.20}$$

since v and e_P commute with y. Since ξ is a separating vector for M, $uy = yu$ and so $u \in P' \cap M = \mathcal{Z}(P)$. The separating property of ξ also ensures that u is unique. $\square$

The *coupling constant* of a nondegenerate action of a II_1 factor M on a Hilbert space $\mathcal{H}$ dates back to Murray and von Neumann [66, Definition 3.3.1] and was subsequently used by Jones to define the index of a subfactor [47]. Given a nonzero vector $\eta \in \mathcal{H}$, write e_η^M for the projection in M' onto the subspace $\overline{M\eta}$ and $e_\eta^{M'}$ for the projection in M onto $\overline{M'\eta}$. If M' is also a finite factor, with faithful normalized trace $\tau_{M'}$, then the quantity $\dim_M(\mathcal{H}) = \tau_M(e_\eta^{M'})/\tau_{M'}(e_\eta^M)$ is independent of the choice of nonzero $\eta \in \mathcal{H}$ (see [33, Part III Chapter 6] for example) and referred to as the *coupling constant* of M on $\mathcal{H}$ or the *M-dimension of* $\mathcal{H}$. When M' is a II_∞ factor, set $\dim_M(\mathcal{H}) = \infty$. We recall some properties of the M-dimension from [33, Part III Chapter 6].

Properties 2.3.3. *Let M be a II_1 factor acting nondegenerately on $\mathcal{H}$.*

(i) If p' is a projection in M', then $\dim_{Mp'}(p'\mathcal{H}) = \tau_{M'}(p')\dim_M(\mathcal{H})$.
(ii) If p is a projection in M, then $\dim_{pMp}(p\mathcal{H}) = \dim_M(\mathcal{H})/\tau_M(p)$.
(iii) M acts in standard position on $\mathcal{H}$ if and only if $\dim_M(\mathcal{H}) = 1$.
(iv) If $\dim_M(\mathcal{H}) \geq 1$, then there is a tracial vector $\xi \in \mathcal{H}$ for M, in the sense that $\tau_M(x) = \langle x\xi, \xi\rangle$ for all $x \in M$.
(v) If $\dim_M(\mathcal{H}) \leq m \in \mathbb{N}$, then there is a set of m-cyclic vectors for M on $\mathcal{H}$.

2.4. **Some approximation estimates.** We need some standard results for approximating unitaries and projections in the sequel. Lemma 2.4.1 is a consequence of [56, Lemma 1.10] (it follows by noting that the function $\alpha(t)$ used there satisfies $\alpha(t) \leq \sqrt{2}t$ for $0 \leq t < 1$), and Lemma 2.4.2 is the usual estimate in the Murray-von Neumann equivalence of close projections (see [64, Lemma 6.2.1] for example).

Lemma 2.4.1. *Suppose that $\gamma < 1$ and let $M \subset_\gamma N$ be a near inclusion of von Neumann algebras sharing the same unit.*

(i) Given a unitary $u \in M$, there exists a unitary $v \in N$ with $\|u - v\| < \sqrt{2}\gamma$.
(ii) Given a projection $p \in M$, there exists a projection $q \in N$ with $\|p - q\| < 2^{-1/2}\gamma$.

Lemma 2.4.2. *Let p and q be projections in a von Neumann algebra M with $\|p - q\| < 1$. Then there exists a unitary $u \in M$ with $upu^* = q$ and $\|u - I_M\| \leq \sqrt{2}\|p - q\|$.*

Lemma 3.6 of [26] examined the center-valued traces on close finite von Neumann algebras. The next lemma gives improved estimates in the special case of close II_1 factors.

Lemma 2.4.3. *Let M and N be II_1 factors acting nondegenerately on a Hilbert space $\mathcal{H}$ and satisfying $M \subset_\gamma N \subset_\gamma M$ for a constant $\gamma < 2^{-3/2}$. Let $P \subseteq M \cap N$ be a diffuse von Neumann algebra containing $I_\mathcal{H}$. Then $\tau_M|_P = \tau_N|_P$.*

Proof. Let $p_1 \in P$ satisfy $\tau_M(p_1) = 1/2$, and choose a unitary $v \in M$ so that $vp_1v^* = I_\mathcal{H} - p_1$. By Lemma 2.4.1 (i) we may choose a unitary $u \in N$ so that $\|v - u\| \leq \sqrt{2}\gamma$. Then

$$\|vp_1v^* - up_1u^*\| \leq 2\|v - u\| \leq 2\sqrt{2}\gamma < 1 \tag{2.21}$$

by the choice of the bound on γ. Thus up_1u^* and $I_\mathcal{H} - p_1$ are equivalent projections in N so $\tau_N(p_1) = \tau_N(up_1u^*) = \tau_N(I_\mathcal{H} - p_1)$, proving that $\tau_N(p_1) = 1/2$.

Now consider $p_2 \in P$ with $\tau_M(p_2) = 1/2^2$ and choose $p_1 \in P$ so that $p_2 \leq p_1$ and $\tau_M(p_1) = 1/2$. Since $\tau_N(p_1) = 1/2$, the normalized traces on p_1Mp_1 and p_1Np_1 are respectively $2\tau_M$ and $2\tau_N$. Applying the argument above to $p_2 \in p_1Pp_1 \subseteq (p_1Mp_1) \cap (p_1Np_1)$, noting that $p_1Mp_1 \subset_\gamma p_1Np_1 \subset_\gamma p_1Mp_1$, gives $\tau_M(p_2) = \tau_N(p_2) = 1/2^2$. We may continue inductively to obtain that $\tau_M(p_n) = \tau_N(p_n)$ for any projection $p_n \in P$ with $\tau_M(p_n) = 1/2^n$, $n \geq 1$. The span of such projections is weak*-dense in P, so the result follows from normality of the traces. □

Lemma 2.4.4. *Let M and N be* II_1 *factors acting nondegenerately on a Hilbert space $\mathcal{H}$ with $M \subset_\gamma N$ for some $\gamma < 1$. Let Φ be a state on $\mathcal{B}(\mathcal{H})$ extending τ_N. Then*

$$|\tau_M(x) - \Phi(x)| \leq (2 + 2\sqrt{2})\gamma\|x\| < 5\gamma\|x\|, \quad x \in M \setminus \{0\}. \tag{2.22}$$

Proof. Fix x in the unit ball of M and a unitary $u \in M$. Choose $y \in N$ with $\|x - y\| \leq \gamma$ and, by Lemma 2.4.1 (i), a unitary $v \in N$ with $\|u - v\| < \sqrt{2}\gamma$. Then

$$\begin{aligned} \|uxu^* - vyv^*\| &= \|uxu^* - vxv^* + v(x - y)v^*\| \\ &\leq \|v^*ux - xv^*u\| + \|x - y\| \\ &\leq 2\|v^*u - I_{\mathcal{H}}\| + \|x - y\| \\ &= 2\|u - v\| + \|x - y\| \leq (2\sqrt{2} + 1)\gamma, \end{aligned} \tag{2.23}$$

and so

$$\begin{aligned} |\Phi(uxu^*) - \Phi(x)| &\leq |\Phi(uxu^*) - \Phi(vyv^*)| + |\Phi(vyv^*) - \Phi(y)| + |\Phi(y) - \Phi(x)| \\ &< (1 + 2\sqrt{2})\gamma + \gamma = (2 + 2\sqrt{2})\gamma < 5\gamma, \end{aligned} \tag{2.24}$$

since $\Phi(vyv^*) = \tau_N(vyv^*) = \tau_N(y) = \Phi(y)$. The Dixmier approximation theorem (see [53, Theorem 8.3.5]) shows that $\tau_M(x)I_{\mathcal{H}}$ is a norm limit of convex combinations of elements of the form uxu^*. The estimate (2.22) follows immediately. □

In a similar vein to the previous lemma, close inclusions inside II_1 factors have close relative commutants (see [19, Proposition 2.7] which states Lemma 2.4.5 when $M = N$). Recall that if P is a unital von Neumann subalgebra of a II_1 factor M, then the unique τ_M-preserving conditional expectation $E^M_{P' \cap M}$ from M onto $P' \cap M$ satisfies the condition that $E_{P' \cap M}(x)$ is the unique element of minimal $\|\cdot\|_2$-norm in the $\|\cdot\|_2$-norm closure of the convex set $\mathrm{co}\{uxu^* : u \in \mathcal{U}(P)\} \subseteq L^2(M)$ for each $x \in M$ (see [96, Lemma 3.6.5 (i)], for example).

Lemma 2.4.5. *Let M and N be* II_1 *factors acting nondegenerately on a Hilbert space $\mathcal{H}$ and suppose that $P \subseteq M$ and $Q \subseteq N$ are unital von Neumann subalgebras.*

(i) Suppose that $M \subseteq_\gamma N$ and $Q \subseteq_\delta P$. Then $P' \cap M \subseteq_{2\sqrt{2}\delta + \gamma} Q' \cap N$.

(ii) Suppose that $M \subseteq_{cb,\gamma} N$ and $Q \subseteq_\delta P$. Then $P' \cap M \subseteq_{cb,2\sqrt{2}\delta + \gamma} Q' \cap N$.

Proof. (i). Given $x \in P' \cap M$ with $\|x\| \leq 1$ choose $y \in N$ with $\|x - y\| \leq \gamma$. For a unitary $v \in Q$ use Lemma 2.4.1 (i) to find a unitary $u \in P$ with $\|u - v\| \leq \sqrt{2}\delta$. Noting that $uxu^* = x$, we obtain the estimate

$$\begin{aligned} \|vyv^* - x\| &\leq \|vyv^* - vxv^*\| + \|vxv^* - uxu^*\| \\ &\leq \|y - x\| + 2\|v - u\| \leq \gamma + 2\sqrt{2}\delta. \end{aligned} \tag{2.25}$$

Since $E_{Q' \cap N}(y)$ is a strong operator limit of convex combinations of elements vyv^* for unitaries $v \in Q$, it follows that $\|E_{Q' \cap N}(y) - x\| \leq 2\sqrt{2}\delta + \gamma$, as required.

(ii). Fix $n \in \mathbb{N}$. We have $P \otimes \mathbb{C}I_n \subseteq M \otimes \mathbb{M}_n$ and $Q \otimes \mathbb{C}I_n \subseteq N \otimes \mathbb{M}_n$ and the near inclusions $M \otimes \mathbb{M}_n \subseteq_\gamma N \otimes \mathbb{M}_n$, $P \otimes \mathbb{C}I_n \subseteq_\delta Q \otimes \mathbb{C}I_n$. By part (i) we have

$$(P' \cap M) \otimes \mathbb{M}_n = (P \otimes \mathbb{C}I_n)' \cap (M \otimes \mathbb{M}_n) \subseteq_{\gamma + 2\sqrt{2}\delta} (Q \otimes \mathbb{C}I_n)' \cap (N \otimes \mathbb{M}_n) = (Q' \cap N) \otimes \mathbb{M}_n. \tag{2.26}$$

Since n was arbitrary, the completely bounded near inclusion $P' \cap M \subseteq_{cb,2\sqrt{2}\delta + \gamma} Q' \cap N$ holds. □

2.5. **Group cohomology and crossed products.** Recall that if $\alpha : \Gamma \curvearrowright X$ is an action of a countable discrete group Γ on an abelian group X, then we have a cochain complex $C^n(\Gamma, X) = \{f : \Gamma^n \to X\}$ with coboundary map $\partial : C^n(\Gamma, X) \to C^{n+1}(\Gamma, X)$ given by

$$\begin{aligned} \partial(f)(g_0, \dots, g_n) =& \alpha_{g_0}(f(g_1, \dots, g_n)) + \sum_{i=0}^{n-1} (-1)^{i+1} f(g_0, \dots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \dots, g_n) \\ & + (-1)^{n+1} f(g_0, \dots, g_{n-1}), \quad f \in C^n(\Gamma, X),\ g_0, \dots, g_n \in \Gamma, \end{aligned} \tag{2.27}$$

inducing the cohomology groups $H^\bullet(\Gamma, X)$. These are defined as follows. The map $\phi \in C^n(\Gamma, X)$ is an n-cocycle if $\partial\phi = 0$, while it is an n-coboundary if there exists $\psi \in C^{n-1}(\Gamma, X)$ so that $\phi = \partial\psi$. The spaces of n-cocycles and n-coboundaries are denoted respectively $Z^n(\Gamma, X)$ and $B^n(\Gamma, X)$, and the n^{th} cohomology group is defined as $Z^n(\Gamma, X)/B^n(\Gamma, X)$. When X is a normed vector space, the *bounded cochains*

$$C_b^n(\Gamma, X) = \{f \in C^n(\Gamma, X) : \|f\| = \sup_{g_1, \dots, g_n \in \Gamma} \|f(g_1, \dots, g_n)\| < \infty\} \tag{2.28}$$

define a subcomplex of $C^\bullet(\Gamma, X)$ which gives rise to the bounded cohomology groups $H_b^\bullet(\Gamma, X)$. When X is a Banach space and the bounded cohomology group $H_b^n(\Gamma, X)$ vanishes, the open mapping theorem shows that there exists a constant $K > 0$, with the following property:

$$\forall \psi \in Z_b^n(\Gamma, X),\ \exists \phi \in C_b^{n-1}(\Gamma, X) \text{ such that } \psi = \partial\phi \text{ and } \|\phi\| \le K\|\psi\|. \tag{2.29}$$

The cohomology groups of relevance to this paper arise from the action $\alpha : \Gamma \curvearrowright P$ of a countable discrete group Γ on a von Neumann algebra P with a separable predual. This restricts to an action of Γ on the abelian group $\mathcal{U}(\mathcal{Z}(P))$ of unitaries in the center of P and so we can consider $H^2(\Gamma, \mathcal{U}(\mathcal{Z}(P)))$. Concretely, a function $\omega : \Gamma \times \Gamma \to \mathcal{Z}(\mathcal{U}(P))$ is a 2-*cocycle* if

$$\alpha_g(\omega(h, k))\omega(gh, k)^*\omega(g, hk)\omega(g, h)^* = I_P, \quad g, h, k \in \Gamma. \tag{2.30}$$

Two 2-cocycles ω and ω' are cohomologous if they differ by a coboundary, in the sense that there exists $\nu : \Gamma \to \mathcal{U}(\mathcal{Z}(P))$ with

$$\omega'(g, h) = \alpha_g(\nu(h))\nu(gh)^*\nu(g)\omega(g, h), \quad g, h \in \Gamma. \tag{2.31}$$

The second cohomology group $H^2(\Gamma, \mathcal{U}(\mathcal{Z}(P)))$ consists of the equivalence classes of the group $Z^2(\Gamma, \mathcal{U}(\mathcal{Z}(P)))$ of 2-cochains under the relation of being cohomologous. Recall that a 2-cocycle ω is *normalized* if $\omega(g, h)$ is trivial when either $g = e$ or $h = e$ and every 2-cocycle is cohomologous to a normalized 2-cocycle, so there is no loss of generality in restricting to normalized cocycles.

Given a 2-cocycle $\omega \in Z^2(\Gamma, \mathcal{U}(\mathcal{Z}(P)))$, the twisted crossed product von Neumann algebra $P \rtimes_{\alpha,\omega} \Gamma$ is a von Neumann algebra generated by a unital copy of P and unitaries $(u_g)_{g \in \Gamma}$ for which there is a faithful normal conditional expectation $E : P \rtimes_{\alpha,\omega} \Gamma \to P$ and for which the conditions

$$u_g x u_g^* = \alpha_g(x), \quad x \in P,\ g \in \Gamma, \tag{2.32}$$

$$u_g u_h = \omega(g, h) u_{gh}, \quad g, h \in \Gamma, \tag{2.33}$$

$$E(u_g) = 0, \quad g \in \Gamma,\ g \neq e, \tag{2.34}$$

hold. If in addition ω is normalized, then u_e is the identity operator. The crossed product is usually constructed concretely starting from a faithful representation of P, but for our purposes all that matters is that these algebras are characterized by conditions (2.32) and

(2.34) above with the cocycle being obtained by (2.33) (see [15, 100] for example). The isomorphism class of the crossed product only depends on the cohomology class of the cocycle ω. We sketch these well known facts below.

Proposition 2.5.1. *Let $\alpha : \Gamma \curvearrowright P$ be a trace-preserving action of a countable discrete group on a finite von Neumann algebra P with a separable predual and let $\omega \in Z^2(\Gamma, \mathcal{U}(\mathcal{Z}(P)))$.*

(i) *Suppose that a finite von Neumann algebra M is generated by a unital copy of P and unitaries $(u_g)_{g\in\Gamma}$ with $u_e = I_P$ and there is a faithful normal trace-preserving expectation $E_P^M : M \to P$ so that (2.32), (2.34) hold and $u_g u_h \in \mathcal{Z}(P) u_{gh}$ for $g, h \in \Gamma$ (this last condition is automatic if $P' \cap M \subseteq P$). Then M is *-isomorphic to $P \rtimes_{\alpha,\omega} \Gamma$ where ω is a normalized 2-cocycle given by (2.33). Further, an isomorphism can be found which identifies the two copies of P and maps the unitaries u_g in M to the canonical unitaries in the crossed product $P \rtimes_{\alpha,\omega} \Gamma$.*

(ii) *If $\omega, \omega' \in Z^2(\Gamma, \mathcal{U}(\mathcal{Z}(P)))$ are cohomologous, then $P \rtimes_{\alpha,\omega} \Gamma$ and $P \rtimes_{\alpha,\omega'} \Gamma$ are *-isomorphic.*

Sketch of the proof. (i). Note that by (2.32) $u_h u_g x u_g^* u_h^* = u_{gh} x u_{gh}^*$ for all $x \in P$ so that $u_{gh}^* u_g u_h \in P' \cap M$. Thus if $P' \cap M \subseteq P$, then $u_g u_h \in \mathcal{Z}(P) u_{gh}$ for all $g, h \in \Gamma$.

Now let $\mathcal{U}_0$ be the subgroup of $\mathcal{U}(M)$ generated by the u_g's and $\mathcal{U}(\mathcal{Z}(P))$. Then $\mathcal{U}(\mathcal{Z}(P))$ is normal in $\mathcal{U}_0$. Define $\omega(g,h) = u_g u_h u_{gh}^*$ which lies in $\mathcal{U}(\mathcal{Z}(P))$ by the assumption that $u_g u_h \in \mathcal{Z}(P) u_{gh}$. This last assumption, the fact that $u_e = I_P$ and condition (2.34) can be used to show that for $g \in \Gamma$ the cosets $u_g \mathcal{U}(\mathcal{Z}(P))$ are distinct. Thus the quotient $\mathcal{U}_0/\mathcal{U}(\mathcal{Z}(P))$ can be canonically identified with Γ giving rise to the extension

$$1 \to \mathcal{U}(\mathcal{Z}(P)) \hookrightarrow \mathcal{U}_0 \xrightarrow{q} \Gamma \to 1 \tag{2.35}$$

which induces the action α on $\mathcal{U}(\mathcal{Z}(P))$ by (2.32). Since $g \mapsto u_g$ is a set theoretic right inverse of the quotient map q, it follows that ω is a normalised 2-cocycle (see [6, Chapter IV]).

Now represent M in standard position on $L^2(M)$ and $P \rtimes_{\alpha,\omega} \Gamma$ in standard position on $L^2(P) \otimes \ell^2(\Gamma)$. The relation (2.32) shows that the collection M_0 of finite sums $\sum_{g\in\Gamma} x_g u_g$ with $x_g \in P$ is dense in M. Further, as E_P^M is a trace-preserving conditional expectation satisfying (2.34), the map $U_0 : M_0 \to P \rtimes_{\alpha,\omega} \Gamma$ given by $U_0(\sum_{g\in\Gamma} x_g u_g) = \sum_{g\in\Gamma} x_g v_g$ (where $(v_g)_{g\in\Gamma}$ are the canonical unitaries in $P \rtimes_{\alpha,\omega} \Gamma$) is 2-norm isometric. Thus we can extend U_0 to a unitary operator $U : L^2(M) \to L^2(P) \otimes \ell^2(\Gamma)$. It is then readily checked that $UMU^* = P \rtimes_{\alpha,\omega} \Gamma$ and $\mathrm{Ad}(U)$ has the desired properties.

(ii). Suppose that $\omega, \omega' \in Z^2(\Gamma, \mathcal{U}(\mathcal{Z}(P)))$ are cohomologous via $\nu : \Gamma \to \mathcal{U}(\mathcal{Z}(P))$ satisfying (2.31). Let $(u_g)_{g\in\Gamma}$ be the canonical unitaries in $P \rtimes_{\alpha,\omega} \Gamma$ and define $v_g = \nu(g) u_g \in \mathcal{U}(P \rtimes_{\alpha,\omega} \Gamma)$. It is easy to see that $P \rtimes_{\alpha,\omega} \Gamma$ is generated by $(P \cup \{v_g : g \in \Gamma\})$, $v_g x v_g^* = \alpha_g(x)$ for $x \in P$, $E_P(v_g) = 0$ for $g \neq e$ and $v_g v_h v_{gh}^* = \omega'(g,h)$. By the previous part $P \rtimes_{\alpha,\omega} \Gamma = P \rtimes_{\alpha,\omega'} \Gamma$. □

If the the unitary-valued cocycles used to define twisted crossed products contain a common spectral gap, then we can take a logarithm to obtain bounded cocycles. The easy lemma below collects this fact for later use.

Lemma 2.5.2. *Suppose that $\alpha : \Gamma \curvearrowright P$ is a trace preserving action of a discrete group on a finite von Neumann algebra P with a separable predual. Let $\omega \in Z^2(\Gamma, \mathcal{U}(\mathcal{Z}(P)))$ satisfy $\sup_{g,h\in\Gamma} \|\omega(g,h) - I_P\| < \sqrt{2}$.*

(i) The expression $\psi = -i\log\omega$ defines a bounded cocycle in $Z_b^2(\Gamma, \mathcal{Z}(P)_{sa})$.
(ii) If $\psi = \partial\phi$ for some $\phi \in C^1(\Gamma, \mathcal{Z}(P)_{sa})$, then $\omega = \partial\nu$, where $\nu(g) = e^{i\phi(g)}$ is a 1-cochain in $C^1(\Gamma, \mathcal{U}(\mathcal{Z}(P)))$.
(iii) Regarding ψ as a cocycle taking values in $L^2(\mathcal{Z}(P))_{sa}$, if $\psi = \partial\phi$ for some $\phi \in C^1(\Gamma, L^2(\mathcal{Z}(P))_{sa})$, then $\omega = \partial\nu$ for $\nu(g) = e^{i\phi(g)}$.

Proof. (i). Let $\log : \mathbb{T} \setminus \{-1\} \to i\mathbb{R}$ be the continuous logarithm with $\log(1) = 0$. For each $g, h \in \Gamma$, the spectrum of $\omega(g,h)$ lies in the domain of log, so we can define $\psi(g,h) = -i\log(\omega(g,h)) \in \mathcal{Z}(P)_{sa}$. Note that $\|\psi\| \leq \pi/2$. For $g,h,k \in \Gamma$, the spectra of the operators $\alpha_g(\omega(h,k))$ and $\omega(g,hk)$ are contained in $\{z \in \mathbb{T} : |z-1| < \sqrt{2}\} = \{e^{i\theta} : -\pi/2 < \theta < \pi/2\}$ so that

$$\log(\alpha_g(\omega(h,k))\omega(g,hk)) = \log(\alpha_g(\omega(h,k))) + \log(\omega(g,hk)). \tag{2.36}$$

Approximating the logarithm function by polynomials, we have

$$\log(\alpha_g(\omega(h,k))) = \alpha_g(\log(\omega(h,k))) \tag{2.37}$$

and this leads to the equation

$$\alpha_g(\log(\omega(h,k))) + \log(\omega(g,hk)) = \log(\alpha_g(\omega(h,k))\omega(g,hk))). \tag{2.38}$$

Similarly, $\log(\omega(gh,k)) + \log(\omega(g,h)) = \log(\omega(gh,k)\omega(g,h))$, so ψ satisfies the cocycle identity

$$\alpha_g(\psi(h,k)) - \psi(gh,k) + \psi(g,hk) - \psi(g,h) = 0, \quad g,h,k \in \Gamma, \tag{2.39}$$

and so defines an element of $Z_b^2(\Gamma, \mathcal{Z}(P)_{sa})$.

(ii). When $\psi = \partial\phi$ for some $\phi \in C^1(\Gamma, \mathcal{Z}(P)_{sa})$, we have $\partial\exp(i\phi) = \exp(i\partial\phi) = \omega$ as $\mathcal{Z}(P)$ is abelian.

(iii). This is similar to (ii). Regard ψ as an element of $Z^2(\Gamma, L^2(\mathcal{Z}(P))_{sa})$. If we have $\psi = \partial\phi$ for some $\phi \in C^1(\Gamma, L^2(\mathcal{Z}(P)_{sa}))$, then $\omega = e^{i\partial\phi} = \partial\exp(i\phi)$, noting that exponentiating an element of $L^2(\mathcal{Z}(P))$ in this fashion produces a unitary element of $\mathcal{Z}(P)$. $\square$

We will be most interested in those crossed products M which are II_1 factors and the subalgebra P satisfies $P' \cap M \subseteq P$. There are two well known sets of conditions which give rise to this situation.

(1) The twisted version of Murray and von Neumann's classical group-measure space construction from [65]. We take $P = L^\infty(X,\mu)$ to be an abelian von Neumann algebra and the action $\alpha : \Gamma \curvearrowright P$ to arise from an ergodic, probability measure preserving action $\Gamma \curvearrowright (X,\mu)$. The resulting twisted crossed product $P \rtimes_{\alpha,\omega} \Gamma$ is a II_1 factor under the additional hypothesis that the action is essentially free (meaning that the set $\{x \in X : g \cdot x = x\}$ is μ-null for all $g \in \Gamma$, $g \neq e$). In this case, P is a maximal abelian subalgebra of $P \rtimes_{\alpha,\omega} \Gamma$ (see the discussion of crossed products in [53]).
(2) When P is a II_1 factor and $\alpha : \Gamma \curvearrowright P$ is an outer action (in the sense that α_g is not inner for $g \neq e$), then the crossed product $P \rtimes_\alpha \Gamma$ is again a II_1 factor and P is an irreducible subfactor of the crossed product, [68].

To unify these situations, say that an action $\alpha : \Gamma \curvearrowright P$ is *properly outer* if, for $g \in \Gamma$ with $g \neq e$ and a nonzero projection $z \in \mathcal{Z}(P)$ with $\alpha_g(z) = z$, the automorphism of Pz induced by α_g is not inner. If the fixed point algebra of the restriction of α to $\mathcal{Z}(P)$ is trivial, then we say that the action is *centrally ergodic*. When P is abelian, these conditions reduce to freeness and ergodicity, while if $\mathcal{Z}(P) = \mathbb{C}1$ they reduce to outerness. It is folklore that the twisted crossed products produced by trace preserving actions satisfying these conditions give rise to II_1 factors M satisfying the key relation $P' \cap M \subseteq P$.

Proposition 2.5.3. *Let P be a finite von Neumann algebra with a fixed faithful normal normalized trace τ_P and suppose that $\alpha : \Gamma \curvearrowright P$ is a trace preserving, centrally ergodic, and properly outer action of a countable discrete group Γ. Then, for any $\omega \in Z^2(\Gamma, \mathcal{U}(\mathcal{Z}(P)))$, the twisted crossed product $M = P \rtimes_{\alpha,\omega} \Gamma$ is a II_1 factor and $P' \cap M \subseteq P$.*

2.6. **Normalizers.** Normalizers were introduced by Dixmier [32] in order to to distinguish maximal abelian subalgebras (masas) up to automorphisms of the larger algebra. We will use them to study the structure of twisted crossed products.

Definition 2.6.1. Given an inclusion $P \subseteq M$ of von Neumann algebras, the *normalizers* of P in M consist of those unitaries $u \in M$ with $uPu^* = P$. These form a subgroup of the unitary group of M which we denote $\mathcal{N}(P \subseteq M)$. The subalgebra P is called *singular* if $\mathcal{N}(P \subseteq M) = \mathcal{U}(P)$ and *regular* if $\mathcal{N}(P \subseteq M)'' = M$.

Twisted crossed products provide the prototype of regular inclusions (the algebra P is always regular in $P \rtimes_{\alpha,\omega} \Gamma$) but not all regular inclusions arise from a properly outer action in this way. In [38], Feldman and Moore introduced the notion of a *Cartan* masa: a masa $A \subseteq M$ is *Cartan* if it is regular and there exists a faithful normal conditional expectation from M onto A. Cartan masas arise from measurable equivalence relations via a generalized crossed product construction. We will return to this construction in Subsection 6.4, where we describe a version of our main results in this setting.

Given a II_1 factor M and a regular von Neumann subalgebra P containing I_M, a *bounded homogenous orthonormal basis of normalizers* (see [43, Definition 4.1]) for $P \subseteq M$ is a family $(u_n)_{n \geq 0}$ in $\mathcal{N}(P \subseteq M)$ such that $u_0 = I_M$, $E_P^M(u_i^* u_j) = \delta_{i,j} I_M$ for all $i, j \geq 0$ and $\sum_{n=0}^{\infty} u_n P$ is dense in $L^2(M)$. Note that the condition $E_P^M(u_i^* u_j) = 0$ for $i \neq j$ is equivalent to $\overline{u_i P} \perp \overline{u_j P}$ in $L^2(M)$. Again, the prototypical behavior is found in the twisted crossed product construction where the canonical unitaries $(u_g)_{g \in G}$ implementing the action provide a bounded homogeneous orthornormal basis of normalizers. More generally, such a basis can always be found when P is a Cartan masa in M [43, Lemma 4.2].

We end this section by recording a result about close normalizers from [52] in a form suitable for later use.

Proposition 2.6.2. *Let $P \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra and suppose that $u_1, u_2 \in \mathcal{N}(P \subseteq \mathcal{B}(\mathcal{H}))$ satisfy $\|u_1 - u_2\| < 1$. Then there exist unitaries $v \in P$ and $v' \in P'$ with $u_2 = u_1 v v'$, $\|v - I_{\mathcal{H}}\| \leq \sqrt{2}\|u_1 - u_2\|$, and $\|v' - I_{\mathcal{H}}\| \leq (\sqrt{2} + 1)\|u_1 - u_2\|$.*

Proof. The automorphism $\theta = \mathrm{Ad}(u_1^* u_2)$ of P has $\|\theta - \mathrm{id}_P\| \leq 2\|u_1 - u_2\| < 2$. By [52, Lemma 5], there is a unitary $v \in P$ with $\mathrm{Ad}(v) = \theta$ whose spectrum is contained in the half plane

$$\Re(z) \geq \frac{1}{2}\left(4 - \|\theta - \mathrm{id}_P\|^2\right)^{1/2}. \tag{2.40}$$

Computing $\|v - I_P\|$ via the spectral radius gives

$$\begin{aligned} \|v - I_P\|^2 &\leq \left(1 - \frac{1}{2}\left(4 - \|\theta - \mathrm{id}_P\|^2\right)^{1/2}\right)^2 + 1 - \left(\frac{1}{2}\left(4 - \|\theta - \mathrm{id}_P\|^2\right)^{1/2}\right)^2 \\ &\leq 2 - (4 - \|\theta - \mathrm{id}_P\|^2)^{1/2} \leq \|\theta - \mathrm{id}_P\|^2/2 \leq 2\|u_1 - u_2\|^2. \end{aligned} \tag{2.41}$$

Since $v^*u_1^*u_2 \in P'$, we can write $u_2 = u_1vv'$ for a unitary $v' \in P'$ with $\|v' - I_{P'}\| \leq \|v - I_P\| + \|u_1 - u_2\| \leq (1+\sqrt{2})\|u_1 - u_2\|$. ☐

2.7. **Vanishing results for bounded cohomology.** Given a properly outer, centrally ergodic, trace preserving action $\alpha : \Gamma \curvearrowright P$ of a countable discrete group on a finite von Neumann algebra with separable predual, it will be of interest to know when the bounded cohomology groups $H_b^2(\Gamma, \mathcal{Z}(P)_{sa})$ vanish. Examples of nonamenable groups Γ all of whose actions have this property are given by the work of Burger, Monod and Shalom. We are grateful to Monod for explaining to us in [62] how the results of [9, 63, 61] can be combined to provide such groups. Our presentation below is based on [62].

Recall from [60] (noting that in our situation Γ is discrete so issues of continuity disappear) that a Banach Γ-module is a Banach space equipped with an action π of G by isometries. The dual action on the dual Banach space E^* of a Γ-module E gives a $\Gamma^{\rm op}$-module (the dual action reverses the order of multiplication). The identification $g \mapsto g^{-1}$ of Γ with $\Gamma^{\rm op}$ followed by π^* gives a Γ-module structure on E^*, called the contragredient of E. A *coefficient* Γ-module is the contragredient of a separable Banach Γ-module (the choice of predual forms part of the data). Morphisms between coefficient modules are those Γ-equivariant maps arising from Γ-equivariant maps at the level of preduals. A coefficient module V is called *semiseparable* [61, Definition 3.11] if there is a separable coefficient module U and an injective map $V \to U$ of coefficient modules. In particular, given a trace preserving, centrally ergodic action $\alpha : \Gamma \curvearrowright P$ of a discrete group on a finite von Neumann algebra with separable predual, the Γ-module $\mathcal{Z}(P)_{sa}$ is the contragredient of the action $\alpha : G \curvearrowright L^1(\mathcal{Z}(P)_{sa})$ and so is a coefficient module. Further, the module $\mathcal{Z}(P)_{sa}$ is semiseparable, as the canonical embedding $\mathcal{Z}(P)_{sa} \hookrightarrow L^2(\mathcal{Z}(P))_{sa}$ is the contragredient of $L^2(\mathcal{Z}(P)_{sa}) \to L^1(\mathcal{Z}(P)_{sa})$. The direct sum decomposition of $\mathcal{Z}(P)_{sa}$ as $\mathbb{R}I_P \oplus \mathcal{Z}(P)^0_{sa}$, where $\mathcal{Z}(P)^0_{sa} = \{z \in \mathcal{Z}(P)_{sa} : \tau_P(z) = 0\}$ and Γ acts trivially on $\mathbb{R}I_P$, is a direct sum decomposition as semi-separable coefficient modules and so $H_b^2(\Gamma, \mathcal{Z}(P)_{sa}) = 0$ if and only if $H_b^2(\Gamma, \mathcal{Z}(P)^0_{sa}) = 0$ and $H_b^2(\Gamma, \mathbb{R}) = 0$.

Let k be a local field (for example $\mathbb{R}$ or a non-archimedian local field such as a finite extension of the p-adic numbers). Let G be a connected, almost k-simple algebraic group over k with rank at least 2 and let Γ be a lattice in G. Burger and Monod show in [8, 9] (see the proof of [9, Corollary 24]) that $H_b^2(\Gamma, \mathbb{R})$ is isomorphic to $H_c^2(G, \mathbb{R})$, the continuous cohomology of the underlying group G (in which the cochain complex consists of jointly continuous maps). This last group is known, and vanishes unless $k = \mathbb{R}$ and $\pi_1(G)$ is infinite (this last condition is equivalent to the canonical symmetric space arising from a maximal torus being hermitian). In particular, $H_b^2(\Gamma, \mathbb{R}) = 0$ when $\Gamma = SL_n(\mathbb{Z})$ with $n \geq 3$. This vanishing result can be found explicitly as [63, Theorem 1.4] which shows further that, under the same hypotheses on Γ, $H_b^2(\Gamma, E) = 0$ for all separable coefficient modules E. In particular, this shows that when $\alpha : \Gamma \curvearrowright P$ is a properly outer, centrally ergodic, trace preserving action of such a group Γ on a finite von Neumann algebra P with separable predual and finite dimensional center, then $H_b^2(\Gamma, \mathcal{Z}(P)_{sa}) = 0$.

In order to obtain vanishing results when $\mathcal{Z}(P)_{sa}$ is infinite dimensional, and so nonseparable as a Banach space, we need to use the results of [61] which handle semiseparable modules. Let $G = \prod G_i(k_i)$ be a finite product of connected, simply connected semisimple k_i-groups for local fields k_i. Then, given a lattice Γ in G and a semiseparable coefficient module V for Γ with no invariant vectors, the bounded cohomology groups $H_b^2(\Gamma, V)$ vanish provided the minimal rank of each k_i-almost simple factor of G_i is at least 2 for every i (see [61, Corollary 1.8], and [61, Corollary 1.6] for the special case of a lattice in a connected, simply connected, almost simple group). In particular, we can take $V = \mathcal{Z}(P)^0_{sa}$ in this result for any centrally ergodic, properly outer trace preserving action $\alpha : \Gamma \curvearrowright P$ on a finite von Neumann algebra with separable predual.

In the case of $SL_n(\mathbb{Z})$, the previous two paragraphs give the following result.

Theorem 2.7.1 (Monod). *Let $\Gamma = SL_n(\mathbb{Z})$ for $n \geq 3$. Then, for any properly outer centrally ergodic trace preserving action of Γ on a finite von Neumann algebra P with separable predual, the cohomology group $H^2_b(\Gamma, Z(P)_{sa})$ vanishes.*

In the case of irreducible lattices Γ in finite products $G = \prod G_i(k_i)$ of at least 2 factors as above, [9, Corollary 24] shows that $H^2_b(\Gamma, \mathbb{R}) = 0$ when G has no hermitian factors. Just as above, this can be combined with the results of [61] to show that the required bounded cohomology groups vanish. For example, for $n \geq 3$ and a prime p, $SL_n(\mathbb{Z}[1/p])$ is an irreducible lattice in $SL_3(\mathbb{R}) \times SL_3(\mathbb{Q}_p)$, and so $H^2_b(SL_n(\mathbb{Z}[1/p]), Z(P)_{sa}) = 0$ for all properly outer, centrally ergodic, trace preserving actions $SL_n(\mathbb{Z}[1/p]) \curvearrowright P$ on a finite von Neumann algebra with separable predual.

3. Normalizers of amenable subalgebras

Amenable von Neumann algebras were shown to be strongly Kadison-Kastler stable in [18, 89, 44]. In this section, we exploit this result, and its extension to near inclusions [21] to examine the normalizer structure of close inclusions of amenable von Neumann algebras. The situation we will investigate is summarized by the diagram below.

$$\begin{array}{ccc} P & \subseteq & M \\ \wr & & \wr \\ \delta\text{-close} & & \gamma\text{-close} \\ \wr & & \wr \\ Q & \subseteq & N \end{array}$$

Here M and N are γ-close von Neumann algebras on a Hilbert space $\mathcal{H}$ containing von Neumann subalgebras P and Q respectively which are δ-close. Our objective is to transfer normalizers of P in M to normalizers of Q in N. This can be done when P (and hence Q) are amenable or when M (and hence N) are finite (Lemma 3.2.1). Theorem 3.2.3 then gives a canonical isomorphism between $\mathcal{N}(P \subseteq M)/\mathcal{U}(P)$ and $\mathcal{N}(Q \subseteq N)/\mathcal{U}(Q)$ provided γ and δ are sufficiently small and $P' \cap M \subseteq P$. In Section 5.1, we show how these ideas can be used to transfer structural properties like solidity from one II_1 factor to a nearby factor.

3.1. Existing Perturbation results for von Neumann algebras.

We first record the embedding theorems and other existing perturbation results from [19, 21] which we will use repeatedly in the sequel. In part (iii) below, note that the hypothesis of the original statement in [19, Theorem 4.1] is that $d(M, N) < 1/8$, however the proof only needs the hypothesis $M \subset_\gamma N$ and $N \subset_\gamma M$ given here.

Theorem 3.1.1. *(i) (*[21, Theorem 4.3, Corollary 4.4]*) Let P be an amenable von Neumann subalgebra of $\mathcal{B}(\mathcal{H})$ containing $I_{\mathcal{H}}$. Suppose that B is another von Neumann subalgebra of $\mathcal{B}(\mathcal{H})$ and $P \subset_\gamma B$ for a constant $\gamma < 1/100$. Then there exists a unitary $u \in (P \cup B)''$ with $uPu^* \subseteq B$, $\|I_{\mathcal{H}} - u\| \leq 150\gamma$ and $\|uxu^* - x\| \leq 100\gamma\|x\|$ for $x \in P$. If, in addition, $\gamma < 1/101$ and $B \subset_\gamma P$, then $uPu^* = B$.*

(ii) *(*[21, Corollary 4.2]*) If $M, N \subseteq \mathcal{B}(\mathcal{H})$ are amenable von Neumann algebras containing $I_{\mathcal{H}}$ and $M \subset_\gamma N$ for a constant $\gamma < 1/8$ then there is a unitary $u \in (M \cup N)''$ such that $\|I_{\mathcal{H}} - u\| \leq 12\gamma$ and $uMu^* \subseteq N$. Additionally, if $N \subset_\gamma M$ then u may be chosen to also satisfy $uMu^* = N$.*

(iii) *(*[19, Theorem 4.1]*) If P and Q are unital von Neumann subalgebras of a finite von Neumann algebra M satisfying $P \subset_\gamma Q \subset_\gamma P$ for a constant $\gamma < 1/8$ then there is a unitary $u \in (P \cup Q)'' \subseteq M$ with $\|I_M - u\| \leq 7\gamma$ such that $uPu^* = Q$.*

3.2. **Transferring normalizers.** The next lemma embarks on the process of relating normalizers of an inclusion $P \subseteq M$ to those for a nearby inclusion $Q \subseteq N$. Although we will only be using these results for amenable subalgebras, we have stated them in full generality whenever possible.

Lemma 3.2.1. *Let M and N be von Neumann subalgebras of $\mathbb{B}(\mathcal{H})$ containing $I_{\mathcal{H}}$ and satisfying $M \subset_\gamma N \subset_\gamma M$ for a constant $\gamma > 0$. Let $P \subseteq M$ and $Q \subseteq N$ be von Neumann subalgebras satisfying $P \subset_\delta Q \subset_\delta P$ for a constant $\delta \geq 0$.*

(i) Suppose that P and Q are amenable and that $2\delta + \delta^2 + 2\sqrt{2}\gamma < 1/8$. Given $v \in \mathcal{N}(P \subseteq M)$, there exists $v' \in \mathcal{N}(Q \subseteq N)$ with $\|v - v'\| < 25\delta + 25\sqrt{2}\gamma$.

(ii) Suppose that M and N are finite von Neumann algebras, and that $2\delta + \delta^2 + 2\sqrt{2}\gamma < 1/8$. Then the conclusion of (i) holds with the improved estimate $\|v - v'\| < 15\delta + 15\sqrt{2}\gamma$.

(iii) Suppose that P is amenable, that $\gamma < 1/(2\sqrt{2})$ and that $Q = P$. Given $v \in \mathcal{N}(P \subseteq M)$, there exists $v' \in \mathcal{N}(P \subseteq N)$ such that $\|v - v'\| \leq (4 + 2\sqrt{2})\gamma$.

(iv) Suppose that P and Q are amenable. Given $v \in \mathcal{N}(P \subseteq M)$ and $v' \in \mathcal{N}(Q \subseteq N)$ with $\|v - v'\| < 1 - 24\delta$, there exist unitaries $u \in (P \cup Q)''$, $w \in P$ and $w' \in P'$ satisfying $\|u - I_{\mathcal{H}}\| \leq 12\delta$, $\|w - I_{\mathcal{H}}\| < 2^{1/2}(24\delta + \|v - v'\|)$, $\|w' - I_{\mathcal{H}}\| < (2^{1/2} + 1)(24\delta + \|v - v'\|)$, and $u^*v'u = vww'$.

Proof. (i)-(ii). Let $v \in \mathcal{N}(P \subseteq M)$ and choose a unitary $u \in N$ with $\|u - v\| < \sqrt{2}\gamma$ by Lemma 2.4.1 (i). Fix $x \in Q$ with $\|x\| \leq 1$, and choose $y \in P$ so that $\|x - y\| < \delta$, in which case $\|y\| < 1 + \delta$. Then $\|vxv^* - vyv^*\| < \delta$ and $\|uxu^* - vxv^*\| < 2\sqrt{2}\gamma$ so $\|uxu^* - vyv^*\| < \delta + 2\sqrt{2}\gamma$. Since $vyv^* \in P$, there exists $x_1 \in Q$ with $\|vyv^* - x_1\| < \delta(1+\delta)$, and so $\|uxu^* - x_1\| < 2\delta + \delta^2 + 2\sqrt{2}\gamma$. This shows that $uQu^* \subset_{2\delta+\delta^2+2\sqrt{2}\gamma} Q \subset_{2\delta+\delta^2+2\sqrt{2}\gamma} uQu^*$, where the second near inclusion follows by applying the same argument to v^* and u^*.

For (i) we are assuming that P and Q are amenable, so from Theorem 3.1.1 (ii), there exists a unitary $w \in N$ so that $wuQu^*w^* = Q$ and $\|I_{\mathcal{H}} - w\| \leq 12(2\delta + \delta^2 + 2\sqrt{2}\gamma)$. Define $v' = wu \in \mathcal{N}(Q \subseteq N)$. Then

$$\begin{aligned} \|v - v'\| &= \|v - wu\| \\ &\leq \|v - u\| + \|I_{\mathcal{H}} - w\| \\ &\leq 24\delta + 12\delta^2 + 25\sqrt{2}\gamma < 25\delta + 25\sqrt{2}\gamma \end{aligned} \tag{3.1}$$

since $\delta < 1/16$. This proves (i).

For (ii) we assume that M and N are finite von Neumann algebras with no restrictions on P and Q. The counterpart of the unitary w above may now be chosen so that $\|I_{\mathcal{H}} - w\| \leq 14\delta + 7\delta^2 + 14\sqrt{2}\gamma$ using Theorem 3.1.1 (iii), leading to the estimate $\|v - v'\| < 15\delta + 15\sqrt{2}\gamma$. This proves (ii).

(iii). Now suppose that P is amenable and $P = Q$. If $v \in \mathcal{N}(P \subseteq M)$, then Lemma 2.4.1 (i) allows us to choose a unitary $u \in N$ with $\|v - u\| \leq \sqrt{2}\gamma$. Thus $x \mapsto u^*vxv^*u$

defines an isomorphism ϕ of P into N with $\|\phi(x)-x\| \leq 2\|u-v\| \leq 2\sqrt{2}\gamma < 1$ for $x \in P$, $\|x\| \leq 1$. From [18, Theorem 4.2], there exists a unitary $w \in N$ so that $\phi(x) = wxw^*$ and $\|w - I_{\mathcal{H}}\| \leq 4\gamma$. Define $v' = uw \in \mathcal{N}(P \subseteq N)$ and estimate

$$\begin{aligned}\|v-v'\| &= \|v-uw\| = \|w-u^*v\| \\ &\leq \|w-I_{\mathcal{H}}\| + \|I_{\mathcal{H}} - u^*v\| \\ &\leq 4\gamma + \sqrt{2}\gamma = (4+\sqrt{2})\gamma, \end{aligned} \tag{3.2}$$

as required.

(iv). The hypothesis carries an implicit assumption that $\delta < 1/24$, so Theorem 3.1.1 (ii) allows us to choose a unitary $u \in (P \cup Q)''$ with $\|u - I_{\mathcal{H}}\| \leq 12\delta$ and $uPu^* = Q$. Let $u_1 = u^*v'u$ and $u_2 = v$ which both normalize P. Then

$$\begin{aligned}\|u_1-u_2\| &\leq \|u^*v'u - u^*vu\| + \|u^*vu - v\| \\ &\leq \|v'-v\| + 2\|u - I_{\mathcal{H}}\| \leq \|v'-v\| + 24\delta < 1. \end{aligned} \tag{3.3}$$

By Proposition 2.6.2 there exist unitaries $w \in P$ and $w' \in P'$ with $u_1 = u_2ww'$, $\|w - I_{\mathcal{H}}\| \leq \sqrt{2}\|u_1 - u_2\|$ and $\|w' - I_{\mathcal{H}}\| \leq (\sqrt{2}+1)\|u_1-u_2\|$. Thus $u^*v'u = vww'$ with $\|w - I_{\mathcal{H}}\| \leq \sqrt{2}(\|v-v'\| + 24\delta)$ and $\|w' - I_{\mathcal{H}}\| \leq (\sqrt{2}+1)(\|v-v'\|+24\delta)$. □

Lemma 3.2.2. *Let M and N be von Neumann algebras acting nondegenerately on a Hilbert space $\mathcal{H}$ and suppose that $M \subset_\gamma N \subset_\gamma M$ where $\gamma > 0$. Let $P \subseteq M$ and $Q \subseteq N$ be von Neumann subalgebras where $P \subset_\delta Q \subset_\delta P$ for a constant $\delta \geq 0$. Suppose that $7\delta + 3\sqrt{2}\gamma < 1$ and that one of the following statements holds:*

(a) P and Q are both amenable.
(b) M and N are both finite.

Then:

(i) $P' \cap M \subseteq P$ if and only if $Q' \cap N \subseteq Q$;
(ii) P is a masa in M if and only if Q is a masa in N.

Proof. (i). Fix $\varepsilon > 0$ such that $7\delta + 3\sqrt{2}\gamma + \varepsilon < 1$. We will argue by contradiction, so suppose that $P' \cap M \subseteq P$ but that $Q' \cap N$ strictly contains $\mathcal{Z}(Q)$. Since the quotient map of $Q' \cap N$ onto $(Q' \cap N)/\mathcal{Z}(Q)$ has norm 1, we may choose a unitary $u \in Q' \cap N$ so that $\|u - z\| \geq 1 - \varepsilon$ for all $z \in \mathcal{Z}(Q)$. By Lemma 2.4.1 (i), there is a unitary $v \in P$ so that $\|u - v\| < \sqrt{2}\gamma$. Let $w \in P$ be an arbitrary unitary and choose $y \in Q$ such that $\|w - y\| \leq \delta$. Then

$$\begin{aligned}\|wvw^* - v\| &= \|wv - vw\| \leq \|wu - uw\| + 2\sqrt{2}\gamma \\ &\leq \|yu - uy\| + 2\delta + 2\sqrt{2}\gamma = 2\delta + 2\sqrt{2}\gamma \end{aligned} \tag{3.4}$$

since y commutes with u. In case (a), when P is amenable, we average this inequality over a generating amenable unitary subgroup of P to obtain an element $z_P \in P' \cap M = \mathcal{Z}(P)$ so that $\|z_P\| \leq 1$ and $\|z_P - v\| \leq 2\delta + 2\sqrt{2}\gamma$. In case (b), when M is finite, we use the fact that the conditional expectation $E^M_{P'\cap M}(v)$ is a weak*-limit point of convex combinations of the form $\sum_{i=1}^n w_i v w_i^*$ for unitaries $w_i \in P$ to obtain such a $z_P = E^M_{P'\cap M}(v)$ satisfying the same estimates. Now choose $x \in Q$ with $\|x - z_P\| \leq \delta$. For any unitary $w_1 \in Q$, choose $y_1 \in P$

with $\|w_1 - y_1\| \leq \delta$. Then

$$\begin{aligned} \|w_1 x w_1^* - x\| &= \|w_1 x - x w_1\| \leq \|w_1 z_P - z_P w_1\| + 2\delta \\ &\leq \|y_1 z_P - z_P y_1\| + 4\delta = 4\delta \end{aligned} \tag{3.5}$$

since y_1 commutes with z_P. A second averaging argument in Q gives an element $z_Q \in \mathcal{Z}(Q)$ satisfying $\|z_Q - x\| \leq 4\delta$. The identity

$$u - z_Q = (u - v) + (v - z_P) + (z_P - x) + (x - z_Q) \tag{3.6}$$

yields the estimate $\|u - z_Q\| \leq \sqrt{2}\gamma + (2\delta + 2\sqrt{2}\gamma) + \delta + 4\delta = 7\delta + 3\sqrt{2}\gamma$. Since $\|u - z_Q\| \geq 1 - \varepsilon$, we obtain the contradiction $7\delta + 3\sqrt{2}\gamma + \varepsilon \geq 1$, proving (i) since it is symmetric in M and N.

(ii). This will follow from (i) once we know that Q is abelian. In [51, Corollary C] it is shown that algebras within distance $1/10$ of an abelian C^*-algebra are necessarily abelian. The argument below improves the estimate in this result.

Let $u \in Q$ be a unitary and consider an element $x \in Q$ with $\|x\| \leq 1$. Then choose $y_1, y_2 \in P$ with $\|u - y_1\|, \|x - y_2\| \leq \delta$. Since y_1 and y_2 commute and are both bounded in norm by $1 + \delta$, it is easy to estimate that

$$\|uxu^* - x\| = \|ux - xu\| \leq 2\delta + 2\delta(1 + \delta) < 6\delta < 6/7. \tag{3.7}$$

Averaging over Q using either amenability of Q or finiteness of N as in part (i) gives an element $z \in \mathcal{Z}(Q)$ such that $\|z - x\| \leq 6/7$. Thus the quotient map of Q onto $Q/\mathcal{Z}(Q)$ has norm at most $6/7$ and we conclude that $Q = \mathcal{Z}(Q)$ so is abelian. □

We now obtain an isomorphism between the quotient groups $\mathcal{N}(P \subseteq M)/\mathcal{U}(P)$ and $\mathcal{N}(Q \subseteq N)/\mathcal{U}(Q)$ provided $P' \cap M \subseteq P$. Write $[v]$ for the coset of a normalizer $v \in \mathcal{N}(P \subseteq M)$ in the quotient group $\mathcal{N}(P \subseteq M)/\mathcal{U}(P)$.

Theorem 3.2.3. *Let M and N be von Neumann algebras containing $I_{\mathcal{H}}$ on a Hibert space $\mathcal{H}$ satisfying $M \subset_\gamma N$ and $N \subset_\gamma M$ for a positive constant γ. Let $P \subseteq M$ be a von Neumann subalgebra satisfying $P' \cap M \subseteq P$ and let $Q \subseteq N$ be a von Neumann subalgebra such that $P \subset_\delta Q$ and $Q \subset_\delta P$ for some $\delta \geq 0$. Suppose that one of the following statements holds:*
(a) M and N are both finite and

$$2(15\sqrt{2}\gamma + 15\delta) + \sqrt{2}\delta < 1; \tag{3.8}$$

(b) P and Q are both amenable and

$$2(25\sqrt{2}\gamma + 25\delta) + \sqrt{2}\delta < 1. \tag{3.9}$$

Then there is a group isomorphism

$$\Theta : \mathcal{N}(P \subseteq M)/\mathcal{U}(P) \to \mathcal{N}(Q \subseteq N)/\mathcal{U}(Q) \tag{3.10}$$

such that, for each $v \in \mathcal{N}(P \subseteq M)$, $\Theta([v])$ has a representative $w \in \mathcal{N}(Q \subseteq N)$ with

$$\|w - v\| < \begin{cases} 15\sqrt{2}\gamma + 15\delta, & \textit{in case } (a); \\ 25\sqrt{2}\gamma + 25\delta, & \textit{in case } (b). \end{cases} \tag{3.11}$$

Proof. In both case (a) and case (b), Lemma 3.2.2 (i) applies to show that $Q' \cap N \subseteq Q$.

Note that if $w, w' \in \mathcal{N}(Q \subseteq N)$ have $\|w - w'\| < 1$, then Proposition 2.6.2 gives unitaries $q \in Q$ and $q' \in Q'$ with $w' = wqq'$. Since $q' \in Q' \cap N \subseteq Q$, it follows that $[w] = [w']$.

Similarly two normalizers $v, v' \in \mathcal{N}(P \subseteq M)$ with $\|v - v'\| < 1$ represent the same element of $\mathcal{N}(P \subseteq M)/\mathcal{U}(P)$.

Define

$$\alpha = \begin{cases} 15\sqrt{2}\gamma + 15\delta, & M \text{ and } N \text{ are finite;} \\ 25\sqrt{2}\gamma + 25\delta, & \text{otherwise.} \end{cases} \tag{3.12}$$

For each $v \in \mathcal{N}(P \subseteq M)$, Lemma 3.2.1 gives a unitary $w \in \mathcal{N}(Q \subseteq N)$ with $\|v - w\| < \alpha$ (using part (ii) in case (a) and part (i) in case (b)). We shall define Θ by $\Theta([v]) = [w]$, where w is such a normalizer. Given another normalizer $w' \in \mathcal{N}(Q \subseteq N)$ with $\|v - w'\| < \alpha$, we have $\|w - w'\| < 2\alpha < 1$ so that $[w] = [w']$ and the definition of Θ does not depend on the choice of w. Now suppose that $v' \in \mathcal{N}(P \subseteq M)$ has $[v] = [v']$ so $v = v'u$ for some $u \in \mathcal{U}(P)$. By Lemma 2.4.1 (i), there exists a unitary $u_1 \in \mathcal{U}(Q)$ with $\|u - u_1\| < \sqrt{2}\delta$. Let $w, w' \in \mathcal{N}(Q \subseteq N)$ have $\|w - v\|, \|w' - v'\| < \alpha$. Then

$$\|w - w'u_1\| \leq \|w - v\| + \|v'u - w'u\| + \|w'u - w'u_1\| < 2\alpha + \sqrt{2}\delta < 1, \tag{3.13}$$

so that $[w] = [w'u_1] = [w']$. Therefore Θ is well defined.

For $v, v' \in \mathcal{N}(P \subseteq M)$ choose $w, w' \in \mathcal{N}(Q \subseteq N)$ with $\|v - w\|, \|v' - w'\| < \alpha$. Then $\|vv' - ww'\| \leq 2\alpha < 1$ so

$$\Theta([vv']) = [ww'] = [w][w'] = \Theta([v])\Theta([v']), \tag{3.14}$$

showing that Θ is a group homomorphism. Interchanging the roles of M and N, we can define a group homomorphism $\Psi : \mathcal{N}(Q \subseteq N)/\mathcal{U}(Q) \to \mathcal{N}(P \subseteq M)/\mathcal{U}(P)$ by $\Psi([w]) = [v]$ where $v \in \mathcal{N}(P \subseteq M)$ and $w \in \mathcal{N}(Q \subseteq N)$ have $\|v - w\| < \alpha$. In this way Ψ is the inverse of Θ and so Θ is bijective. □

4. Reduction to standard position and Cartan masas

Given two close II_1 factors M and N on a general Hilbert space $\mathcal{H}$, our objective in this section is to show how we can construct close isomorphic copies of these algebras on a new Hilbert space $\mathcal{K}$ so that they both act in standard position on $\mathcal{K}$. Difficulties arise because the usual strategy of changing representations by amplification and compression must be employed in a fashion compatible with both algebras. In particular, as we do not assume that M' and N' are close on $\mathcal{H}$, we must take care in compressing by projections in M'; these need not be close to projections in N'. This suggests that in the cases when M' and N' are already known to be close, then matters are more straightforward. This is correct; we set out the details in Section 4.3.

Once we have reached standard position other tasks become easier. In particular, given an amenable von Neumann subalgebra $P \subseteq M \cap N$ with $P' \cap M \subseteq P$ we can adjust the situation so that the inclusions $P \subseteq M$ and $P \subseteq N$ give the same (not just close) basic construction algebras. It is then straightforward to show that P is regular in M if and only if it is regular in N.

4.1. **Algebras in standard position.** Lemma 3.7 of [26] shows that if M is a finite von Neumann algebra in standard position on the Hilbert space $\mathcal{H}$ and N is another von Neumann algebra on $\mathcal{H}$ close to M such that M' and N' are also close, then N is approximately in standard position in the sense that its coupling function is uniformly close to 1. We start by showing that, working in the factor case for ease of calculation, the assumption that M' and N' are close is unnecessary in [26, Lemma 3.7].

Lemma 4.1.1. *Let M be a* II_1 *factor acting non-degenerately on* $\mathcal{H}$ *with* $\dim_M(\mathcal{H}) \le 1$. *Let* N *be another* II_1 *factor acting on this Hilbert space which satisfies* $M \subset_\gamma N$ *and* $N \subset_\gamma M$ *for a positive constant* γ. *The following statements hold:*

(i) $N' \subset_{2(1+\sqrt{2})\gamma} M'$;
(ii) if $\gamma < 1/22$, *then* N' *is finite;*
(iii) if $\gamma < 1/47$, *then* $M' \subset_{4(1+\sqrt{2})\gamma} N'$.

Proof. (i). Since M has a cyclic vector on $\mathcal{H}$, this is Proposition 2.1.6 (ii).

(ii). Assume that $\gamma < 1/22$ and suppose that N' is infinite. Take $v \in N'$ with $v^*v = I_{\mathcal{H}}$ and $vv^* = e$ for some projection $e \ne I_{\mathcal{H}}$. Then, by part (i), there is an element $w \in M'$ with $\|w - v\| \le 2(1+\sqrt{2})\gamma$, which implies that $\|w\| \le 1 + 2(1+\sqrt{2})\gamma$. Since $v^*v = I_{\mathcal{H}}$, it follows that

$$\begin{aligned}\|w^*w - I_{\mathcal{H}}\| &= \|w^*w - v^*v\| \\ &\le \|w^*w - w^*v\| + \|w^*v - v^*v\| \\ &\le (1 + 2(1+\sqrt{2})\gamma)2(1+\sqrt{2})\gamma + 2(1+\sqrt{2})\gamma \\ &= (2 + 2(1+\sqrt{2})\gamma)(2(1+\sqrt{2})\gamma),\end{aligned} \tag{4.1}$$

and similarly $\|ww^* - vv^*\| \le (2 + 2(1+\sqrt{2})\gamma)(2(1+\sqrt{2})\gamma)$. As $\gamma < 1/22$, it follows that $\|w^*w - I_{\mathcal{H}}\| < 1/2 < 1$ and so w^*w is invertible in M'. Since M' is of type II_1, we can write $w = u|w|$, where u is a unitary. Therefore $ww^* = uw^*wu^*$ also satisfies $\|ww^* - I_{\mathcal{H}}\| = \|u(w^*w - I_{\mathcal{H}})u^*\| < 1/2$. Thus

$$\|e - I_{\mathcal{H}}\| \le \|vv^* - ww^*\| + \|ww^* - I_{\mathcal{H}}\| < \frac{1}{2} + \frac{1}{2} = 1, \tag{4.2}$$

which is a contradiction. This establishes (ii).

(iii). Assume now that $\gamma < 1/47$. Since $\dim_{M'}(\mathcal{H}) \ge 1$, there is a vector $\eta \in \mathcal{H}$ with

$$\tau_{M'}(x) = \langle x\eta, \eta\rangle, \quad x \in M'. \tag{4.3}$$

Now by (i), $N' \subset_{2(1+\sqrt{2})\gamma} M'$ where $2(1+\sqrt{2})\gamma < 1$, and N' is finite by (ii). Thus we may apply Lemma 2.4.4 to this pair to obtain

$$|\tau_{N'}(y) - \langle y\eta, \eta\rangle| < 4(1+\sqrt{2})^2\gamma\|y\|, \quad y \in N'. \tag{4.4}$$

Let $J = \{y \in N' : y\eta = 0\}$. Then J is a weakly closed left ideal in N' so has the form $N'p$ for a projection $p \in N'$. Since $p\eta = 0$, we obtain

$$\tau_{N'}(p) < 4(1+\sqrt{2})^2\gamma < 1/2, \tag{4.5}$$

as $\gamma < 1/47$. Then $\tau_{N'}(I_{\mathcal{H}} - p) > 1/2$, so p is equivalent in N' to a projection $q \le I_{\mathcal{H}} - p$. Choose a unitary $u \in N'$ so that $upu^* = q$ and define $\zeta = u\eta$.

Now suppose that $x \in N'$ satisfies $x\eta = x\zeta = 0$. Then $x \in J$ so $xp = x$ and $x(I_{\mathcal{H}} - p) = 0$. Since $x\zeta = 0$, we have $xu\eta = 0$ so $xu \in J$ and $xu = xup$. Thus, since $q \le I_{\mathcal{H}} - p$,

$$x = xupu^* = xq = x(I_{\mathcal{H}} - p)q = 0. \tag{4.6}$$

This proves that the pair $\{\eta, \zeta\}$ is a separating set for N', and so $\{\eta, \zeta\}$ is a cyclic set for N. From Proposition 2.1.6 (ii), it follows that $M' \subset_{4(1+\sqrt{2})\gamma} N'$ as required. □

With the preceding lemma we can now show that if a II_1 factor M acts in standard position on $\mathcal{H}$, then the same is true for any close algebra N. We work in the context of a distinguished subalgebra A which is a masa in M and in N and satisfies $J_MAJ_M \subseteq N'$. In this case M, M', N and N' can be taken to have exactly the same tracial vector and so we can compare the basic construction algebras obtained from subalgebras of $M \cap N$. In particular the inclusions $A \subseteq M$ and $A \subseteq N$ generate the same basic construction algebras $\langle M, e_A\rangle = \langle N, e_A\rangle$. We will show subsequently that, at the cost of worse numerical estimates, we can reduce to the situation where such a masa A exists.

Lemma 4.1.2. *Let M be a II_1 factor acting in standard position on a Hilbert space $\mathcal{H}$ with tracial vector ξ, and let N be another II_1 factor on $\mathcal{H}$. Suppose that $M \subset_\gamma N$ and $N \subset_\gamma M$ for some constant $\gamma < 1/47$. Suppose further that A is a masa in both M and N and $J_MAJ_M \subseteq N'$. Then the following hold:*

(i) ξ is a tracial vector for both N and N' so that N is also in standard position on $\mathcal{H}$.
(ii) $(M \cup \{e_A\})'' = (J_MAJ_M)' = (J_NAJ_N)' = (N \cup \{e_A\})''$, where J_N is the modular conjugation operator induced from the cyclic and separating vector ξ for N and e_A is the projection onto $\overline{A\xi}$.

Proof. (i). For each unitary $u \in A$ and $y \in N$, we have

$$\langle uyu^*\xi, \xi\rangle = \langle yu^*\xi, u^*\xi\rangle = \langle yJ_MuJ_M\xi, J_MuJ_M\xi\rangle = \langle y\xi, \xi\rangle, \tag{4.7}$$

as $J_MuJ_M \in J_MAJ_M \subseteq N'$. Since $A' \cap N = A$, the unique τ_N-preserving conditional expectation $E^N_A : N \to A$ is given by taking $E^N_A(y)$ to be the unique element of minimal $\|\cdot\|_{2,\tau_N}$-norm in the $\|\cdot\|_{2,\tau_N}$-closed convex hull of $\{uyu^* : u \in \mathcal{U}(A)\}$ (see [96, Lemma 3.6.5], for example). By convexity, (4.7) gives

$$\langle y\xi, \xi\rangle = \langle E^N_A(y)\xi, \xi\rangle = \tau_M(E^N_A(y)), \quad y \in N. \tag{4.8}$$

Lemma 2.4.3 (which applies since $\gamma < 2^{-3/2}$) shows that $\tau_M|_A = \tau_N|_A$. Thus (4.8) gives

$$\tau_N(y) = \tau_N(E^N_A(y)) = \tau_M(E^N_A(y)) = \langle y\xi, \xi\rangle, \quad y \in N, \tag{4.9}$$

so that ξ is a tracial vector for N.

To see that ξ is a tracial vector for N', note that as $\gamma < 1/47$, Lemma 4.1.1 gives $M' \subset_{4(1+\sqrt{2})\gamma} N'$ and $N' \subset_{2(1+\sqrt{2})\gamma} M'$. Since $3\sqrt{2} \times 4(1+\sqrt{2})\gamma < 1$, Lemma 3.2.2 (ii) shows that J_MAJ_M is a masa of N' as J_MAJ_M is a masa of M'. By Lemma 4.1.1 (ii), N' is finite so the estimate $4(1+\sqrt{2})\gamma < 2^{-3/2}$ allows us to apply the previous paragraph to the pair (M', N') to see that ξ is tracial for N', establishing (i).

(ii). Since both M and N are in standard position with respect to the same cyclic and separating vector ξ (though with possibly different modular conjugation operators J_M and J_N), the Hilbert space projection e_A from $\mathcal{H}$ onto $\overline{A\xi}$ used to define the basic construction $\langle M, e_A\rangle$ from the inclusion $A \subseteq M$ is the same projection used to define the basic construction from the inclusion $A \subseteq N$. Now

$$J_MAJ_M \subseteq N' \cap \{e_A\}' = (J_NNJ_N) \cap \{e_A\}' = J_NAJ_N \subseteq J_NNJ_N, \tag{4.10}$$

where we use $e_A = J_Ne_AJ_N = J_Me_AJ_M$ and $A = N \cap \{e_A\}' = M \cap \{e_A\}'$ from Properties 2.3.1 (i) and (ii). As we noted in the previous paragraph, J_MAJ_M is a maximal abelian subalgebra of $N' = J_NNJ_N$, giving the equality $J_MAJ_M = J_NAJ_N$. Taking commutants gives the middle inclusion $(J_MAJ_M)' = (J_NAJ_N)'$ of (ii). The two outermost equalities

$(M\cup\{e_A\})'' = (J_MAJ_M)'$ and $(J_NAJ_N)' = (N\cup\{e_A\})''$ are the characterization of the basic construction algebra given in [47, Proposition 3.1.5 (i)] (see Properties 2.3.1 (iii)). □

Next, we extend the previous result to replace the masa A by an amenable von Neumann subalgebra $P \subseteq M$ with $P'\cap M \subseteq P$ satisfying $J_MPJ_M \subseteq N'$ and aim to show that the basic constructions $\langle M, e_P\rangle$ and $\langle N, e_P\rangle$ are equal. As a consequence, we can show that P is regular in M if and only if it is regular in N. To do this we use the following theorem of Popa from [79, Theorem 3.2]. We quote the version that appears as [96, Theorem 12.2.4] since this records additional information which is implicit in the original.

Theorem 4.1.3 (Popa). *Let P be a von Neumann subalgebra of a finite von Neumann algebra M with separable predual and suppose that $P'\cap M \subseteq P$. If A_0 is a finite dimensional abelian *-subalgebra of P, then there exists a masa A in M such that $A_0 \subseteq A \subseteq P$.*

Lemma 4.1.4. *Let M be a II_1 factor with separable predual acting in standard position on a Hilbert space $\mathcal{H}$ with tracial vector ξ. Let N be a II_1 factor on $\mathcal{H}$ with $M \subset_\gamma N$ and $N \subset_\gamma M$ for some $\gamma < 1/47$. Suppose that P is an amenable subalgebra of $M \cap N$ with $P'\cap M \subseteq P$ and $J_MPJ_M \subseteq N'$. Then the following statements hold.*

(i) N is also in standard position on $\mathcal{H}$ and ξ is a tracial vector for N and N';
(ii) $\langle M, e_P\rangle = (J_MPJ_M)' = (J_NPJ_N)' = \langle N, e_P\rangle$;
(iii) Suppose $P \subseteq M$ is a regular inclusion with a bounded homogeneous basis of normalizers $(u_n)_{n\geq 0}$ (see Section 2.6) and $(v_n)_{n\geq 0}$ is a family of normalizers in $\mathcal{N}(P \subseteq N)$ with $v_0 = I_{\mathcal{H}}$ and satisfying $\|v_n - u_n\| < 1$ for all n. Then $(v_n)_{n\geq 0}$ is a bounded homogeneous basis of normalizers for $P \subseteq N$ and so $(P \cup \{v_n : n \geq 0\})'' = N$.
(iv) $\mathcal{N}(P \subseteq M)$ generates M if and only if $\mathcal{N}(P \subseteq N)$ generates N.

Proof. (i). Since P is amenable and satisfies $P'\cap M \subseteq P$, Lemma 3.2.2 (i) shows that $P'\cap N \subseteq P$. By Theorem 4.1.3 there is a masa A in M with $A \subseteq P$. Then $J_MAJ_M \subseteq J_MPJ_M \subseteq N'$. By Lemma 3.2.2 (ii), A is also a masa in N. Lemma 4.1.2 now shows that N is in standard position on $\mathcal{H}$ with tracial vector ξ and so we can define e_P as the projection onto $\overline{P\xi}$. This projection is used to define the basic constructions $\langle M, e_P\rangle$ and $\langle N, e_P\rangle$.

(ii). Arguing just as in equation (4.10) in Lemma 4.1.2, we have

$$J_MPJ_M \subseteq N'\cap\{e_P\}' = (J_NNJ_N)'\cap\{e_P\}' = J_NPJ_N. \tag{4.11}$$

For the reverse inclusion, suppose first that A_0 is a finite dimensional abelian subalgebra of P and use Theorem 4.1.3 to find a masa A_1 in M with $A_0 \subseteq A_1 \subseteq P$. By Lemma 3.2.2 (ii), A_1 is a masa in N. Since $J_MA_1J_M \subseteq N'$, Lemma 4.1.2 (ii) gives $J_MA_1J_M = J_NA_1J_N$. Thus $J_NA_0J_N \subseteq J_MA_1J_M \subseteq J_MPJ_M$. Fix a self-adjoint operator $x \in P$ and choose a sequence $(x_n)_{n=1}^\infty$ of elements of P which converge to x in the weak operator topology such that each $W^*(x_n)$ is a finite dimensional abelian von Neumann algebra. Applying the previous argument with $A_0 = W^*(x_n)$ gives $J_Nx_nJ_N \in J_MPJ_M$, and so taking weak operator limits, $J_NxJ_N \in J_MPJ_M$. This gives $J_NPJ_N \subseteq J_MPJ_M$ so these algebras are equal. The middle equality in (ii) follows by taking commutants, and the outer two equalities by applying Properties 2.3.1 (iii).

(iii). For each n, apply Proposition 2.6.2 to u_n and v_n to obtain unitaries $w_n \in P$ and $w_n' \in P'$ with $u_n = v_nw_nw_n'$. As $N \subseteq \langle N, e_P\rangle = \langle M, e_P\rangle$, we have $w_n' \in P'\cap\langle M, e_P\rangle$ and so, by Lemma 2.3.2, there exist unitaries $z_n \in \mathcal{Z}(P)$ with $w_n'\xi = z_n\xi$. Thus $u_n\xi = v_nw_nz_n\xi$ and so we have $\overline{u_nP\xi} = \overline{v_nP\xi}$. As noted in Section 2.6, the condition $E_P^M(u_m^*u_n) = 0$ for $m \neq n$ is equivalent to $\overline{u_mP\xi} \perp \overline{u_nP\xi}$. Thus the sequence $(v_n)_{n\geq 0}$ inherits this property and so satisfies $E^N_P(v_m^*v_n) = \delta_{m,n}I$. Further, $\sum_{n=0}^\infty v_nP\xi$ is dense in $\mathcal{H}$ and so $(v_n)_{n=0}^\infty$ is a bounded homogeneous basis of normalizers for $P \subseteq N$. Note that this immediately implies that $(P \cup \{v_n : n \geq 0\})'' = N$. Indeed, if $N_0 \subseteq N$ is the von Neumann algebra generated by P and $(v_n)_{n\geq 0}$, consider $x \in N$ with $E^N_{N_0}(x) = 0$. For each $n \geq 0$ and $b \in P$,

$$0 = \tau_N(b^*v_n^*E^N_{N_0}(x)) = \tau_N(E^N_{N_0}(b^*v_n^*x)) = \tau_N(b^*v_n^*x) = \langle x\xi, v_nb\xi\rangle, \tag{4.12}$$

since ξ is a tracial vector for N. Since $\sum_{n=0}^\infty v_nP\xi$ is dense in $\mathcal{H}$, (4.12) gives $x\xi = 0$, and hence $x = 0$ as ξ is separating for N. Thus $N = N_0$ and $P \cup \{v_n : n \geq 0\}$ generates N.

(iv). The hypotheses of Lemma 3.2.1 (iii) are satisfied so given any normalizer $v \in \mathcal{N}(P \subseteq M)$, there exists $u \in \mathcal{N}(P \subseteq N)$ with $\|v - u\| \leq \alpha$, where $\alpha = (4 + 2\sqrt{2})\gamma$. Using $2\alpha < 1$ and arguing just as in the first paragraph of (iii), we see that $\overline{uP\xi} = \overline{vP\xi}$. Suppose that $\mathcal{N}(P \subseteq M)$ generates M, so that $\mathcal{H} = \overline{\mathrm{span}}\,\{u\xi : u \in \mathcal{N}(P \subseteq M)\}$. Thus $\mathcal{H} = \overline{\mathrm{span}}\,\{v\xi : v \in \mathcal{N}(P \subseteq N)\}$. Just as in the second paragraph of (iii), it then follows that $N = \mathcal{N}(P \subseteq N)''$.

For the reverse implication, we use parts (i) and (ii) to interchange the roles of M and N. We have already noted that $P' \cap N \subseteq P$ and part (ii) shows that $J_NPJ_N = J_MPJ_M \subseteq M'$. Thus if $\mathcal{N}(P \subseteq N)$ generates N, then $\mathcal{N}(P \subseteq M)$ generates M. □

Remark 4.1.5. (i) The hypothesis of a separable predual in Lemma 4.1.4 is present only to enable us to use Theorem 4.1.3 to find a masa in M that lies in P. Consequently it can be dropped if P happens to be a masa in M.

(ii) In the special case that P is a masa in M in Lemma 4.1.4, it follows immediately that P is Cartan in M if and only if it is Cartan in N. Alternatively, this can be read off from Popa's characterization of Cartan and singular masas in terms of the structure of the basic construction. Part (i) of the proposition in [82, Section 1.4.3] uses the equivalence of the algebras generated by normalizers and by quasinormalizers of a masa in [86] to show that a masa B in a II_1 factor Q with a separable predual is Cartan if and only if $B' \cap \langle Q, e_B\rangle$ is generated by finite projections from $\langle Q, e_B\rangle$. Thus, once we know that the masa P satisfies $\langle M, e_P\rangle = \langle N, e_P\rangle$, it follows that it is Cartan in M if and only if it is Cartan in N.

Given close II_1 factors M and N on a Hilbert space $\mathcal{H}$ with a cyclic vector for M, the results of Theorem 3.1.1 combine with Lemma 4.1.2 to show that M and N have the same coupling constant on $\mathcal{H}$. We make the extra assumption that M and N have a common masa, a situation to which we will reduce subsequently.

Proposition 4.1.6. *Suppose that M and N are II_1 factors acting nondegenerately on $\mathcal{H}$ with $M \subset_\gamma N$ and $N \subset_\gamma M$ for $\gamma < 1/136209$. Moreover, suppose that $M \cap N$ contains an abelian von Neumann algebra A which is a masa in M. If $\dim_M(\mathcal{H}) \leq 1$, then $\dim_N(\mathcal{H}) = \dim_M(\mathcal{H})$.*

Proof. Note that A is also a masa in N by Lemma 3.2.2 (ii). Suppose first that $\dim_M(\mathcal{H}) = 1$ so that M is in standard position on $\mathcal{H}$ with a tracial vector ξ for M and M'. Since M is in standard position, Lemma 4.1.1 (iii) gives the near inclusion

$$J_MAJ_M \subseteq J_MMJ_M = M' \subset_{4(1+\sqrt{2})\gamma} N'. \tag{4.13}$$

The bound on γ ensures that $4(1 + \sqrt{2})\gamma < 1/100$, so we can apply Theorem 3.1.1 (i) to obtain a unitary

$$v \in (J_MAJ_M \cup N')'' \subseteq (J_MMJ_M \cup N')'' = (M \cap N)' \subseteq A' \tag{4.14}$$

with $vJ_MAJ_Mv^* \subseteq N'$ and

$$\|v - I_{\mathcal{H}}\| \leq 600(1+\sqrt{2})\gamma, \quad d(v(J_MAJ_M)v^*, J_MAJ_M) \leq 400(1+\sqrt{2})\gamma. \tag{4.15}$$

Define $N_1 = v^*Nv$ so that $J_MAJ_M \subseteq N_1'$. An application of (2.2) gives

$$M \subset_{\gamma_1} N_1, \quad N_1 \subset_{\gamma_1} M \tag{4.16}$$

where $\gamma_1 := (1200(1+\sqrt{2})+1)\gamma$. Since A is a masa in N, it is also a masa in N_1. The initial bound on γ ensures that $\gamma_1 < 1/47$, so we can apply Lemma 4.1.2 (i) to N_1 to conclude that N_1 is in standard position on $\mathcal{H}$. Since N_1 and N are unitary conjugates, we also have $\dim_N(\mathcal{H}) = 1$.

Now suppose that $\dim_M(\mathcal{H}) < 1$. Choose a projection $p \in A$ with $\tau_M(p) = \dim_M(\mathcal{H})$. We can cut by p to obtain

$$pMp \subset_\gamma pNp \subset_\gamma pMp. \tag{4.17}$$

Since pMp is in standard position on $p\mathcal{H}$, the pair pMp and pNp are in the situation of the previous two paragraphs and consequently pNp is in standard position on $p\mathcal{H}$. Thus $\dim_{pNp}(p\mathcal{H}) = 1$, and noting that $\tau_N(p) = \tau_M(p)$ by Lemma 2.4.3, we see that $\dim_N(\mathcal{H}) = \tau_N(p) = \tau_M(p) = \dim_M(\mathcal{H})$ as required. □

Remark 4.1.7. In the situation when M and N are close II_1 factors with close commutants M' and N' on $\mathcal{H}$, applying the previous proposition to one of the pairs (M,N) or (M',N') shows that $\dim_M(\mathcal{H}) = \dim_N(\mathcal{H})$. This happens when M and N are completely close (see Section 4.3). Without assuming that M' and N' are close, one can formulate a version of the previous proposition under the hypothesis that $\dim_M(\mathcal{H}) \leq m$ for $m \geq 1$. However, if we work in this way, then the bound on the maximum size of the near containments to which such a result applies will depend on m. We do not know whether there is some constant $\gamma_0 > 0$ such that any II_1 factors M and N on a Hilbert space $\mathcal{H}$ with $d(M,N) < \gamma_0$ have the same coupling constant. If this were the case, then weak Kadison-Kastler stability would imply Kadison-Kastler stability for II_1 factors.

4.2. **Changing representations to standard position.** The next lemma is designed to handle the situation when $\dim_{\mathcal{H}}(M)$ is large. It enables us to cut M by a projection $e \in M'$ which almost lies in N' such that Me has a cyclic vector for $e\mathcal{H}$ and so $\dim_{Me}(e\mathcal{H}) \leq 1$. Once we have replaced N by a small unitary perturbation N_1 of N so that $e \in N_1'$, the results of the previous subsection will apply to the pair M and N_1' on $e\mathcal{H}$. Note that we do not require a finiteness hypothesis on the von Neumann algebras below.

Lemma 4.2.1. *Let M and N be von Neumann algebras acting nondegenerately on a Hilbert space $\mathcal{H}$ with $M \subset_\gamma N$ and $N \subset_\gamma M$ for a constant $\gamma > 0$. Given a unit vector $\zeta \in \mathcal{H}$, there exists a nonzero subprojection $e \in M'$ of the projection with range $\overline{M\zeta}$ satisfying*

$$\mathrm{dist}\,(e, N') \leq 6(1+\sqrt{2})\gamma + 2((1+\sqrt{2})\gamma)^{1/2}, \tag{4.18}$$

and if M and N are II_1 factors then e may be chosen with the additional property that $\dim_{Me}(e\mathcal{H}) = 1/n$ for an integer n. Moreover, if γ satisfies $\gamma < 1/87$ then there exists a projection $f \in N'$ and a unitary $u \in (M' \cup N')''$ so that

$$\|e - f\| \leq 12(1+\sqrt{2})\gamma + 4((1+\sqrt{2})\gamma)^{1/2}, \quad \|u - I_{\mathcal{H}}\| \leq \sqrt{2}\|e - f\|, \tag{4.19}$$

and $ueu^* = f$.

Proof. Choose $\gamma' < \gamma$ which satisfies $M \subseteq_{\gamma'} N$ and $N \subseteq_{\gamma'} M$. Fix a unit vector $\zeta \in \mathcal{H}$, let $p \in M'$ be the projection onto $\overline{M\zeta}$ and let $q \in N'$ be the projection onto $\overline{N\zeta}$. Given x in the unit ball of M, choose $y \in N$ with $\|x - y\| \leq \gamma'$. Since p commutes with x and q commutes with y, we have the algebraic identity

$$(px)pqp - pqp(px) = p(x-y)qp + pq(y-x)p, \tag{4.20}$$

leading to the estimate

$$\|(px)pqp - pqp(px)\| \leq 2\|x-y\| \leq 2\gamma', \tag{4.21}$$

so$\|\mathrm{ad}(pqp)|_{Mp}\| \leq 2\gamma'$. As the vector ζ is cyclic for Mp acting on the Hilbert space $p\mathcal{H}$, Proposition 2.1.6 (i) combines with (4.21) to give an element $z \in (Mp)' = pM'p$ satisfying

$$\|z - pqp\| \leq 2(1+\sqrt{2})\gamma'. \tag{4.22}$$

After replacing z by $(z+z^*)/2$ if necessary, we may assume that z is self-adjoint. Let $e_1 \in pM'p$ be the spectral projection of z for the interval $[1-2(1+\sqrt{2})\gamma, 1+2(1+\sqrt{2})\gamma']$. Then $z(I_{p\mathcal{H}} - e_1) \leq (1-2(1+\sqrt{2})\gamma)I_{p\mathcal{H}}$. If $e_1\zeta = 0$ then, since $pqp\zeta = \zeta$, we have a contradiction from

$$\begin{aligned} 1 = \langle pqp\zeta, \zeta\rangle &= \langle (pqp - z)\zeta, \zeta\rangle + \langle z(I_{p\mathcal{H}} - e_1)\zeta, \zeta\rangle \\ &\leq 2(1+\sqrt{2})\gamma' + (1-2(1+\sqrt{2})\gamma) < 1, \end{aligned} \tag{4.23}$$

as $\gamma' < \gamma$. Thus $e_1\zeta \neq 0$. From the functional calculus, $\|ze_1 - e_1\| \leq 2(1+\sqrt{2})\gamma$, and so

$$\begin{aligned} \|e_1 - e_1qe_1\| &\leq 2(1+\sqrt{2})\gamma + \|ze_1 - e_1qe_1\| \\ &= 2(1+\sqrt{2})\gamma + \|e_1(z-pqp)e_1\| \leq 4(1+\sqrt{2})\gamma. \end{aligned} \tag{4.24}$$

There are now two cases to consider. If M is not a II_1 factor then rename e_1 as e, omitting the next step. However, if M is a II_1 factor then $\dim_{Me_1}(e_1\mathcal{H}) \leq 1$ since $e_1\zeta$ is a cyclic vector for Me_1 on $e_1\mathcal{H}$. Choose an integer n so that $n\dim_{Me_1}(e_1\mathcal{H}) \geq 1$ and choose a projection $e \in e_1M'e_1$ so that $\tau_{e_1M'e_1}(e) = (n\dim_{Me_1}(e_1\mathcal{H}))^{-1}$. By Properties 2.3.3 (i), $\dim_{Me}(e\mathcal{H}) = \tau_{e_1M'e_1}(e)\dim_{Me_1}(e_1\mathcal{H}) = 1/n$, and

$$\|e - eqe\| = \|e(e_1 - e_1qe_1)e\| \leq \|e_1 - e_1qe_1\| \leq 4(1+\sqrt{2})\gamma \tag{4.25}$$

from (4.24). This shows that (4.25) is satisfied in both cases. Thus, from (4.25),

$$\|e(I_{\mathcal{H}} - q)e\| \leq 4(1+\sqrt{2})\gamma \tag{4.26}$$

so

$$\|(I_{\mathcal{H}} - q)e\| \leq (4(1+\sqrt{2})\gamma)^{1/2}, \tag{4.27}$$

and from this we deduce that

$$\|(I_{\mathcal{H}} - q)e(I_{\mathcal{H}} - q)\| \leq 4(1+\sqrt{2})\gamma. \tag{4.28}$$

Writing

$$e = qeq + qe(I_{\mathcal{H}} - q) + (I_{\mathcal{H}} - q)eq + (I_{\mathcal{H}} - q)e(I_{\mathcal{H}} - q), \tag{4.29}$$

we obtain the estimate

$$\begin{aligned}\|e-qeq\| &\le \|qe(I_{\mathcal{H}}-q)+(I_{\mathcal{H}}-q)eq\|+4(1+\sqrt{2})\gamma\\ &= \max\{\|qe(I_{\mathcal{H}}-q)\|,\|(I_{\mathcal{H}}-q)eq\|\}+4(1+\sqrt{2})\gamma\\ &\le (4(1+\sqrt{2})\gamma)^{1/2}+4(1+\sqrt{2})\gamma. \end{aligned}\tag{4.30}$$

If x is in the unit ball of N, choose $y \in M$ with $\|x-y\| \le \gamma$. Noting that x commutes with q and y commutes with both e and p, the algebraic identity

$$\begin{aligned}xq(qeq) = qxeq &= q(x-y)eq+qyeq = q(x-y)eq+qeyq\\ &= q(x-y)eq+qe(y-x)q+qeqxq\end{aligned}\tag{4.31}$$

gives the inequalities $\|xq(qeq)-(qeq)xq\| \le 2\gamma$ and $\|\mathrm{ad}(qeq)|_{Nq}\| \le 2\gamma$. Since ζ is a cyclic vector for Nq on $q\mathcal{H}$, Proposition 2.1.6 (i) provides an element $t \in (Nq)' = qN'q$ so that $\|t-qeq\| \le 2(1+\sqrt{2})\gamma$, so

$$\|e-t\| \le \|e-qeq\|+\|t-qeq\| \le 2((1+\sqrt{2})\gamma)^{1/2}+6(1+\sqrt{2})\gamma, \tag{4.32}$$

establishing (4.18).

Define $\gamma_1 = 2((1+\sqrt{2})\gamma)^{1/2}+6(1+\sqrt{2})\gamma$, and now suppose that the inequality $\gamma < 1/87$ holds, which ensures that $\gamma_1 < 1/2$. Replacing t by $(t+t^*)/2$ if necessary, we may assume that t is self-adjoint. The Hausdorff distance between the spectra $\mathrm{Sp}(e)$ and $\mathrm{Sp}(t)$ is at most $\|e-t\| \le \gamma_1$ (see for example [31, Proposition 2.1]), and so $\mathrm{Sp}(t)$ is contained in $[-\gamma_1,\gamma_1] \cup [1-\gamma_1, 1+\gamma_1]$. If f denotes the spectral projection of t for the second of these intervals, then $\|f-t\| \le \gamma_1$, giving the estimate $\|e-f\| \le 2\gamma_1 < 1$. In this case Lemma 2.4.2 provides a unitary $u \in (M' \cup N')''$ with $\|u-I_{\mathcal{H}}\| \le \sqrt{2}\|e-f\|$ and $ueu^* = f$. This establishes (4.19). □

Remark 4.2.2. The previous lemma enables us to find a nonzero projection $e \in M'$ close to N' such that Me has a cyclic vector for $e(\mathcal{H})$. However, we have not been able to choose e so that $\tau_{M'}(e)$ is close to $\tau_{M'}(p)$. To see what the difficulty is, suppose that M is a II_1 factor and ζ is a tracial vector for Mp and $pM'p$ on $p(\mathcal{H})$ so that $\tau_{M'}(p) = 1$. Performing the calculation in (4.23) more precisely leads to $\langle ze_1\zeta,\zeta\rangle \ge 2(1+\sqrt{2})(\gamma-\gamma')$ and hence

$$\tau_{M'}(e_1) = \langle e_1\zeta,\zeta\rangle \ge \frac{1}{1+2(1+\sqrt{2})\gamma'}\langle ze_1\zeta,\zeta\rangle \ge \frac{2(1+\sqrt{2})(\gamma-\gamma')}{1+2(1+\sqrt{2})\gamma'}. \tag{4.33}$$

Thus our methods only allow us to control the trace of the projection e_1 (and hence e) in terms of the 'gap' between γ and γ'.

We are now in a position to combine the previous results and show that, starting with two close II_1 factors on a Hilbert space, it is possible to produce new close representations of these factors on another Hilbert space so that both are simultaneously in standard position. This enables us to transfer regular amenable subalgebras from one factor to its close counterpart. The lemma below does this in a form designed for immediate use in Section 6. We then set out versions of this result in a general setting and give some applications to the structure of close factors.

Lemma 4.2.3. *Let M and N be II_1 factors with separable preduals acting nondegenerately on the Hilbert space $\mathcal{H}$ and suppose that there is a positive constant $\gamma < 1.74\times 10^{-13}$ such that $M \subset_\gamma N$ and $N \subset_\gamma M$. Suppose further that there is an amenable von Neumann subalgebra*

$P \subseteq M \cap N$ satisfying $P' \cap M \subseteq P$. Then there exists a separable Hilbert space $\mathcal{K}$ and faithful normal representations $\pi : M \to \mathcal{B}(\mathcal{K})$ and $\rho : N \to \mathcal{B}(\mathcal{K})$ with the following properties:

(i) *$\pi(M)$ and $\rho(N)$ are in standard position on $\mathcal{K}$, and there is a common tracial vector ξ for $\pi(M)$, $\pi(M)'$, $\rho(N)$ and $\rho(N)'$.*

(ii) *$\pi(M) \subset_\beta \rho(N)$ and $\rho(N) \subset_\beta \pi(M)$ for a constant β satisfying the estimate*

$$\beta < 50948\gamma^{1/2} < 1/47.$$

(iii) *If $x \in M$ and $y \in N$ satisfy $\|x\|, \|y\| \leq 1$ and $\|x - y\| \leq \delta$ for some $\delta > 0$, then $\|\pi(x) - \rho(y)\| \leq \delta + \beta$.*

(iv) *$\pi|_P = \rho|_P$.*

(v) *$P' \cap N \subseteq P$.*

(vi) *P is regular in M if and only if it is regular in N.*

(vii) *The basic construction algebras on $\mathcal{K}$ given by $\langle \pi(M), e_{\pi(P)} \rangle = (\pi(M) \cup \{e_{\pi(P)}\})''$ and $\langle \rho(N), e_{\pi(P)} \rangle = (\rho(N) \cup \{e_{\pi(P)}\})''$ are equal, where $e_{\pi(P)}$ is the projection from $\mathcal{K}$ onto $\overline{\pi(P)\xi}$.*

(viii) *If $P \subseteq M$ has a bounded homogeneous orthonormal basis of normalizers $(u_n)_{n\geq 0}$ in $\mathcal{N}(P \subseteq M)$ and $(v_n)_{n\geq 0}$ is a sequence in $\mathcal{N}(P \subseteq N)$ satisfying $\|u_n - v_n\| < 1 - \beta$, then $(v_n)_{n\geq 0}$ is a bounded homogeneous orthonormal basis of normalizers for $P \subseteq N$.*

Proof. Use Popa's Theorem (Theorem 4.1.3) to choose a masa $A \subseteq P$ such that A is also a masa in M. Write $M_1 = M$, $N_1 = N$, $P_1 = P$ and $A_1 = A$. Since $\gamma < 1/87$, we can apply Lemma 4.2.1 to M_1 and N_1 to obtain nonzero projections $e \in M_1'$ and $f \in N_1'$ and a unitary $u_1 \in (M_1' \cup N_1')''$ so that $M_1 e$ has a cyclic vector on $e\mathcal{H}$,

$$\|u_1 - I_{\mathcal{H}}\| \leq \sqrt{2}\|e - f\| \leq \sqrt{2}\left(12(1+\sqrt{2})\gamma + 4((1+\sqrt{2})\gamma)^{1/2}\right) \tag{4.34}$$

and $u_1 e u_1^* = f$. Moreover, we can additionally assume that $\dim_{M_1 e}(e\mathcal{H}) = 1/n$ for an integer n. Since $e \in M_1' \cap (u_1^* N_1 u_1)'$ we can compress these algebras by e. Write $\mathcal{H}_2 = e(\mathcal{H})$, $M_2 = M_1 e$, $N_2 = (u_1^* N_1 u_1)e$, $P_2 = P_1 e$ and $A_2 = A_1 e$ so that M_2 and N_2 act on $\mathcal{H}_2$, $P_2 \subseteq M_2 \cap N_2$ (as u_1 commutes with P_1) and the near inclusions $M_2 \subset_{\gamma_2} N_2 \subset_{\gamma_2} M_2$ hold, where γ_2 is given by

$$\gamma_2 = 2\sqrt{2}\left(12(1+\sqrt{2})\gamma + 4((1+\sqrt{2})\gamma)^{1/2}\right) + \gamma < 17.58\gamma^{1/2}. \tag{4.35}$$

By construction $\dim_{M_2}(\mathcal{H}_2) = 1/n$.

Now define $\mathcal{H}_3 = \mathcal{H}_2 \otimes \mathbb{C}^n$, $M_3 = (M_2 \otimes I_n)$, $P_3 = (P_2 \otimes I_n)$, $A_3 = (A_2 \otimes I_n)$, $N_3 = (N_2 \otimes I_n)$ and $\gamma_3 = \gamma_2$. Then $M_3 \subset_{\gamma_3} N_3$ and $N_3 \subset_{\gamma_3} M_3$, $P_3 \subseteq M_3 \cap N_3$ and $\dim_{M_3} \mathcal{H}_3 = 1$ (by Properties 2.3.3 (i)) so that M_3 is in standard position on $\mathcal{H}_3$.

Consider the masa $A_3 \subseteq M_3$. As $\gamma_3 < 1/47$, Lemma 4.1.1 (iii) gives $J_{M_3} M_3 J_{M_3} \subset_{4(1+\sqrt{2})\gamma_3} N_3'$. As $4(1+\sqrt{2})\gamma_3 < 1/100$, another application of Theorem 3.1.1 (i) provides a unitary $u_3 \in (J_{M_3} A_3 J_{M_3} \cup N_3')'' \subseteq P_3'$ so that $\|I_{\mathcal{H}_3} - u_3\| < 600(1+\sqrt{2})\gamma_3$ and $u_3 (J_{M_3} A_3 J_{M_3}) u_3^* \subseteq N_3'$. Define $\mathcal{H}_4 = \mathcal{H}_3$, $M_4 = M_3$, $P_4 = P_3$, $A_4 = A_3$, $N_4 = u_3^* N_3 u_3$, and

$$\gamma_4 = (1200(1+\sqrt{2})+1)\gamma_3 < (1200(1+\sqrt{2})+1)(17.58)\gamma^{1/2} < 50948\gamma^{1/2} < 1/47. \tag{4.36}$$

Then $J_{M_4} A_4 J_{M_4} \subseteq N_4'$, and $P_4 \subseteq M_4 \cap N_4$ since u_3 commutes with P_3. The estimate (2.2) gives the near inclusion $M_4 \subset_{\gamma_4} N_4 \subset_{\gamma_4} M_4$ and then the bound on γ_4 allows us to apply Lemma 3.2.2 (ii) to conclude that A_4 is also a masa in N_4. The hypotheses of Lemma 4.1.2 are now met, from which we see that M_4, M_4', N_4 and N_4' have a common tracial vector.

At each stage of the proof, the various constructions have ensured that the pairs (M_k, M_{k+1}) are canonically isomorphic via compressions, amplifications or unitary conjugation, while the same is true for (N_k, N_{k+1}), $1 \leq k \leq 3$. Now let $\mathcal{K} = \mathcal{H}_4$ and define two isomorphisms $\pi\colon\ M \to M_4$ and $\rho\colon\ N \to N_4$ as the compositions of the isomorphisms constructed above, whereupon $\pi(P) = P_4 \subseteq M_4 \cap N_4 = \pi(M) \cap \rho(N)$. We take β to be γ_4. This establishes parts (i) and (ii). Since all the unitaries u_k used to construct the isomorphisms π and ρ commute with the corresponding P_k, it follows directly that $\pi|_P = \rho|_P$, giving (iv). The estimate of part (iii) also follows from the explicit form of the π and ρ as a composition of unitary conjugations, compressions and amplifications. The latter two operations do not increase distances and the unitaries involved are all close to the identity operator, so repeated application of the triangle inequality gives the specified estimate.

Item (v) is part (i) of Lemma 3.2.2, while the remaining conditions follow from Lemma 4.1.4. Indeed (vii) is part (ii) of Lemma 4.1.4, while part (iv) of Lemma 4.1.4 shows that $\pi(P)$ is regular in $\pi(M)$ if and only if $\rho(P)$ is regular in $\rho(N)$. Since π and ρ are both faithful, condition (vi) follows. For (viii), given (u_n) and (v_n) as in this condition, note that the hypothesis $\|u_n - v_n\| < 1 - \beta$ gives $\|\pi(u_n) - \rho(v_n)\| < 1$ by part (iii). Thus, as $(\pi(u_n))_n$ is a bounded homogeneous orthonormal basis of normalizers for $\pi(P) \subseteq \pi(M)$, Lemma 4.1.4 (iii) shows that $(\rho(v_n))_n$ is a bounded homogeneous orthonormal basis of normalizers for $\rho(P) \subseteq \rho(N)$. As ρ is faithful, we again deduce that $(v_n)_n$ provides a bounded homogeneous orthonormal basis of normalizers for $P \subseteq N$. $\square$

The procedure above enables us to transfer close II_1 factors on a Hilbert space to another space so that they both act in standard position. Thus for the weakest version Kadison-Kastler stability problem for II_1 factors, we can assume that all factors act in standard position: we record this below. Note that more care is required for the stronger spatial versions of the problem. Once we transfer algebras so that they both lie in standard position, any isomorphism between them will be spatially implemented on the new Hilbert space, but, without further information we do not know whether this isomorphism is spatially implemented on the original Hilbert space.

Theorem 4.2.4. *Let M and N be II_1 factors with separable preduals nondegenerately represented on a Hilbert space $\mathcal{H}$ with $M \subset_\gamma N$ and $N \subset_\gamma M$ for some $\gamma < 5.7 \times 10^{-16}$. Then there exists a Hilbert space $\mathcal{K}$ and faithful normal representations $\pi : M \to \mathcal{B}(\mathcal{K})$ and $\rho : N \to \mathcal{B}(\mathcal{K})$ such that $\pi(M)$, $\pi(M)'$, $\rho(N)$ and $\rho(N)'$ have a common tracial vector on $\mathcal{K}$ and $\pi(M) \subset_\beta \rho(N) \subset_\beta \pi(M)$ for*

$$\beta = 50948 \times (301)^{1/2}\gamma^{1/2} + 300\gamma < 8.84 \times 10^5\gamma^{1/2}. \tag{4.37}$$

Further, given $x \in M$ and $y \in N$ with $\|x\|, \|y\| \leq 1$ and $\|x - y\| \leq \delta$, we have $\|\pi(x) - \rho(y)\| \leq \beta + \delta + 300\gamma$.

Proof. Choose a masa A in M. By Theorem 3.1.1 (i) there is a unitary $u \in (A \cup N)''$ with $\|u - I_{\mathcal{H}}\| \leq 150\gamma$ and $uAu^* \subseteq N$. Consider $N_1 = u^*Nu$, which has $M \subset_{\gamma_1} N_1 \subset_{\gamma_1} M$, where $\gamma_1 = 301\gamma$. As $\gamma_1 < 1.74 \times 10^{-13}$, we can take M and N_1 in Lemma 4.2.3 to obtain representations π and ρ_1 satisfying the properties of that lemma (with the β of that lemma being given by $50948 \times (301)^{1/2}\gamma^{1/2}$). Define $\rho(y) = \rho_1(u^*yu)$. It is routine to verify that π and ρ satisfy the required estimates. $\square$

Since the breakthrough paper [82], there has been considerable interest in how many Cartan masas a II_1 factor contains, up to unitary conjugacy: [70] gives the first class of factors with a unique Cartan masa up to unitary conjugacy; [29] provides the first examples of factors with two Cartan masas which are not even conjugate by an automorphism; [71] provides more examples of factors with unique Cartan masas and new factors with at least two Cartan masas and recently large classes of crossed products have been shown to have unique Cartan masas [14, 87, 88]. At the other end of the spectrum, [99] provides a II_1 factor with unclassifiably many Cartan masas up to conjugacy by an automorphism. Here we show that close II_1 factors have the same Cartan masa structure.

Given a II_1 factor M, let $\mathrm{Cartan}(M)$ be the collection of equivalence classes of Cartan masas in M under the relation $A_1 \sim A_2$ if and only if there is a unitary $u \in M$ with $uA_1u^* = A_2$.

Theorem 4.2.5. *Let M and N be II_1 factors with separable preduals acting nondegenerately on a Hilbert space $\mathcal{H}$ with $M \subset_\gamma N$ and $N \subset_\gamma M$ for a constant $\gamma < 5.7 \times 10^{-16}$.*

(i) Suppose $P \subseteq M$ is an amenable regular von Neumann subalgebra with $P' \cap M \subseteq P$ and $Q \subseteq N$ is a von Neumann subalgebra with $P \subset_\delta Q \subset_\delta P$ for some $\delta \geq 0$ such that $300\gamma + \delta < 1/8$. Then Q is regular in N and satisfies $Q' \cap N \subseteq Q$.

(ii) If A is a Cartan masa in M, then there exists a Cartan masa B in N satisfying $d(A, B) < 100\gamma$.

(iii) There exists a canonical bijective map $\Theta : \mathrm{Cartan}(M) \to \mathrm{Cartan}(N)$, given by $\theta([A]) = [B]$ where $A \subseteq M$ and $B \subseteq N$ are Cartan masas with $d(A, B) < 100\gamma$.

(iv) If M has a unique Cartan masa up to unitary conjugacy, then the same is true for N.

Proof. (i). Since $\gamma < 1/100$, we may apply Theorem 3.1.1 (i) to obtain a unitary $u \in (P \cup N)''$ satisfying $\|u - I_{\mathcal{H}}\| \leq 150\gamma$, $uPu^* \subseteq N$, and $d(P, uPu^*) \leq 100\gamma$. Define $N_1 = u^*Nu$, so that $P \subseteq M \cap N_1$ and $M \subset_{\gamma_1} N_1 \subset_{\gamma_1} M$ where $\gamma_1 = 301\gamma$. Then the bound on γ gives $\gamma_1 < 1.74 \times 10^{-13}$, so we may apply Lemma 4.2.3 to conclude that P is regular in N_1 and $P' \cap N_1 \subseteq P$. Thus $Q_1 := uPu^*$ is regular in N and satisfies $Q_1' \cap N \subseteq Q_1$. Now by equation (2.2), $Q_1 \subset_\eta Q \subset_\eta Q_1$ where $\eta = 300\gamma + \delta < 1/8$. By Theorem 3.1.1 (iii), Q and Q_1 are unitarily conjugate inside N. Thus Q inherits the desired properties from Q_1.

(ii). Given a Cartan masa A in M, there exists a unitary $u \in (A \cup N)''$ with $B = uAu^* \subseteq N$ and $d(A, B) < 100\gamma$ by Theorem 3.1.1 (i). Then B is a masa in N by Lemma 3.2.2 (ii) and then is Cartan by (i).

(iii). From (ii), to each Cartan masa A in M we may associate a Cartan masa B in N so that $d(A, B) < 100\gamma$. Let A_1 be another Cartan masa in M and choose a Cartan masa B_1 in N with $d(A_1, B_1) < 100\gamma$. If there exists a unitary $u \in M$ such that $A_1 = uAu^*$, then by Lemma 2.4.1 (i), there is a unitary $v \in N$ with $\|u - v\| < \sqrt{2}\gamma$. Then

$$\begin{aligned} d(B_1, vBv^*) &< d(B_1, uBu^*) + 2\|u - v\| \\ &< d(B_1, uAu^*) + 2\sqrt{2}\gamma + 100\gamma \\ &= d(B_1, A_1) + (100 + 2\sqrt{2})\gamma < (200 + 2\sqrt{2})\gamma < 1/8. \end{aligned} \tag{4.38}$$

Thus B_1 and vBv^* are unitarily conjugate in N by Theorem 3.1.1 (iii) so B_1 and B are unitarily conjugate in N. This shows that there is a well defined map $\Theta : \mathrm{Cartan}(M) \to \mathrm{Cartan}(N)$, defined on $[A]$ by choosing a Cartan masa B as above and letting $\Theta([A]) = [B]$. In the same way there is a map $\Phi : \mathrm{Cartan}(N) \to \mathrm{Cartan}(M)$ so that for each Cartan masa B in N, $\Phi([B]) = [A]$ where $A \subseteq M$ is chosen so that $d(B, A) < 100\gamma$. By construction Φ is the inverse of Θ so Θ is bijective.

(iv). This is immediate from (iii). $\square$

Remark 4.2.6. When M and N are sufficiently close II_1 factors, we can also transfer other structural properties of a masa $A \subseteq M$ to a sufficiently close masa $B \subseteq N$. Theorem 3.2.3 shows that $B \subseteq N$ is singular if and only if $A \subseteq M$ is singular. Within the class of singular masas, the Pukánszky invariant has perhaps been the most successful method of distinguishing nonconjugate masas (see [96, Chapter 7] for background on this invariant, including its definition). As the Pukánszky invariant of $A \subseteq M$ is defined in terms of the relative commutant of the basic construction, $A' \cap \langle M, e_A \rangle$, it follows from Lemma 4.2.3 that if $A \subseteq M$ and $B \subseteq N$ are sufficiently close masas, then they have the same Pukánszky invariant.

4.3. **The reduction procedure for completely close algebras.** Since completely close II_1 factors, have (completely) close commutants (Proposition 2.2.3), the process of changing representations is much easier in this context. We start by noting that completely close II_1 factors always have the same coupling constant.

Proposition 4.3.1. *Let M and N be II_1 factors acting nondegenerately on a separable Hilbert space $\mathcal{H}$ and suppose that $d_{cb}(M,N) < \gamma$ where*

$$\gamma < (301 \times 136209)^{-1}. \tag{4.39}$$

Then $\dim_M(\mathcal{H}) = \dim_N(\mathcal{H})$.

Proof. If both dimensions are infinite then there is nothing to prove, so suppose without loss of generality that $\dim_M(\mathcal{H}) < \infty$. We have $M \subset_{cb,\gamma} N$ and $N \subset_{cb,\gamma} M$ so these near inclusions are also valid with M' and N' replacing M and N respectively, by Proposition 2.2.3. By symmetry, we may suppose that $\dim_M(\mathcal{H}) \leq 1$, otherwise apply the following argument to M'.

Choose a masa $A \subseteq M$. Since $\gamma < 1/100$, Theorem 3.1.1 (i) gives a unitary $u \in (A \cup N)''$ so that $\|u - I_{\mathcal{H}}\| \leq 150\gamma$, $d(A, uAu^*) \leq 100\gamma$, and $uAu^* \subseteq N$. Define $N_1 = u^*Nu$ and note that $A \subseteq M \cap N_1$ while $d(M, N_1) \leq d(M,N) + 2\|u - I_{\mathcal{H}}\| \leq 301\gamma$. Since $3\sqrt{2} \times 301\gamma < 1$, Lemma 3.2.2 (ii) allows us to conclude that A is also a masa in N_1. The hypotheses of Proposition 4.1.6 are now met since $301\gamma < 1/136209$ and so $\dim_{N_1}(\mathcal{H}) = \dim_M(\mathcal{H})$. The result follows since $\dim_{N_1}(\mathcal{H}) = \dim_N(\mathcal{H})$ by the unitary conjugacy of N and N_1. $\square$

We can also easily change representations of completely close von Neumann algebras.

Proposition 4.3.2. *Let M and N be von Neumann algebras acting nondegenerately on a Hilbert space $\mathcal{H}$ with $d_{cb}(M,N) < \gamma$ for $\gamma < 1$. Given any unital normal *-representation π of M on another Hilbert space $\mathcal{K}$, there exists a unital normal *-representation ρ of N on $\mathcal{K}$ such that $d_{cb}(\pi(M), \rho(N)) \leq 3\gamma$ and $\rho|_{M\cap N} = \pi|_{M\cap N}$. Further, if $x \in M$ and $y \in N$ are contractions with $\|x - y\| \leq \delta$ for some $\delta \geq 0$, then $\|\pi(x) - \rho(y)\| \leq 2\gamma + \delta$.*

Proof. The general theory of normal *-representations [33, I §4 Theorem 3] allows us to choose a set S and a projection $p \in M' \overline{\otimes} \mathcal{B}(\ell^2(S))$ so that π is unitarily equivalent to the *-representation $\pi_1 : x \mapsto (x \otimes I_{\ell^2(S)})p$ of M on $\mathcal{K}_1 = p(\mathcal{K})$. Identifying M and N with their amplifications $M \otimes I_{\ell^2(S)}$ and $N \otimes I_{\ell^2(S)}$ respectively and, noting that $M' \overline{\otimes} \mathbb{B}(\ell^2(S)) \subset_\gamma N' \overline{\otimes} \mathcal{B}(\ell^2(S))$ by Properties 2.2.2 (iii), it follows from Lemma 2.4.1 (ii) that there is a projection $q \in N' \overline{\otimes} \mathcal{B}(\ell^2(S))$ with $\|p - q\| < 2^{-1/2}\gamma$. Since $p, q \in (M \cap N)' \overline{\otimes} \mathcal{B}(\ell^2(S))$, Lemma 2.4.2 gives a unitary $u \in (M \cap N)' \overline{\otimes} \mathcal{B}(\ell^2(S))$ such that $\|u - I_{\mathcal{H}\otimes\ell^2(S)}\| < \gamma$ and $upu^* = q$. Define $\rho_1 : N \to \mathcal{B}(\mathcal{K}_1)$ by $\rho_1(x) = u^*(x \otimes I_{\mathcal{H}\otimes\ell^2(S)})up$ for $x \in N$. By construction

$\rho_1|_{M\cap N} = \pi_1|_{M\cap N}$. Further

$$d_{cb}(\pi_1(M), \rho_1(N)) \leq d_{cb}(M, N) + 2\|u - I_{\mathcal{H}\otimes\ell^2(S)}\| < 3\gamma. \tag{4.40}$$

Now choose a unitary $V : \mathcal{K} \to \mathcal{K}_1$ so that $\pi = V^*\pi_1 V$, and let $\rho = V^*\rho_1 V$ so that $d_{cb}(\pi(M), \rho(N)) < 3\gamma$. When $x \in M$ and $y \in N$ are contractions, we have $\|\pi(x) - \rho(y)\| \leq 2\|u - I_{\mathcal{H}\otimes\ell^2(S)}\| + \|x - y\|$. □

In the presence of complete closeness, we also obtain a more direct proof of the key reduction result Lemma 4.2.3 with improved constants. Note that the constant β below is now $O(\gamma)$ as $\gamma \to 0$ whereas it was $O(\gamma^{1/2})$ in the original (c.f. Remark (1) of [11]).

Theorem 4.3.3. *Let M and N be II_1 factors with separable preduals acting nondegenerately on a Hilbert space $\mathcal{H}$ and suppose that there is a constant $\gamma < 1/(903 \times 47)$ such that $M \subset_{cb,\gamma} N$ and $N \subset_{cb,\gamma} M$. Suppose that $P \subseteq M \cap N$ is an amenable von Neumann algebra satisfying $P' \cap M \subseteq P$. Then there exist a separable Hilbert space $\mathcal{K}$ and faithful normal *-representations $\pi : M \to \mathcal{B}(\mathcal{K})$ and $\rho : N \to \mathcal{B}(\mathcal{K})$ such that:*

(i) Property (i) from Lemma 4.2.3 is satisfied.
(ii) $\pi(M) \subset_{cb,\beta} \rho(N)$ and $\rho(N) \subset_{cb,\beta} \pi(M)$ where $\beta = 903\gamma$.
(iii) If $x \in M$ and $y \in N$ are contractions with $\|x - y\| \leq \delta$ for some $\delta \geq 0$, then $\|\pi(x) - \rho(y)\| \leq \delta + 903\gamma$.
(iv) $\pi|_P = \rho|_P$;
(v) Properties (v-viii) from Lemma 4.2.3 are satisfied (with the value of β above).

Proof. By Proposition 4.3.2 we may find faithful representations $\pi : M \to \mathcal{B}(\mathcal{K})$ and $\rho_1 : N \to \mathcal{B}(\mathcal{K})$ which agree on P such that $M_1 = \pi(M)$ is in standard position on $\mathcal{K}$, $M_1 \subset_{cb,3\gamma} \rho_1(N)$, $\rho_1(N) \subset_{cb,3\gamma} M_1$ and $\|\pi_1(x) - \rho_1(y)\| \leq 3\gamma$ whenever $x \in M$ and $y \in N$ are contractions with $\|x - y\| < \gamma$. Write $P_1 = \pi_1(P) = \rho_1(P)$ and $N_1 = \rho_1(N)$ and fix, by Popa's theorem ([79]), a masa $A_1 \subset M_1$ with $A_1 \subseteq P_1$. Since $M_1' \subset_{3\gamma} N_1'$ by Proposition 2.2.3, apply Theorem 3.1.1 (i) to obtain a unitary $u \in (J_{M_1} A_1 J_{M_1} \cup N_1')''$ so that $\|u - I_{\mathcal{K}}\| < 450\gamma$ and $u J_{M_1} A_1 J_{M_1} u^* \subseteq N_1'$. Define $\rho : N \to \mathcal{B}(\mathcal{K})$ by $\rho(y) = u^*\rho_1(y)u$ so that conditions (ii) and (iii) hold. Property (i) from Lemma 4.2.3 follows from Lemma 4.1.2, noting that the estimate on γ ensures that $\beta < 1/47$. Property (v) from Lemma 4.2.3 is now obtained from part (i) of Lemma 3.2.2 while the remaining properties follow from Lemma 4.1.4 in just the same way as in the proof of Lemma 4.2.3. □

5. Structural properties of close II_1 factors

We will now use the methods of Sections 3 and 4 to show that close factors share the same structural properties in the spirit of [26]. A key objective, achieved in Lemma 5.2.4, is to show that if M is a McDuff II_1 factor and N is another factor close to M, then, after making a small unitary perturbation, we can simultaneously factor $M = M_1 \overline{\otimes} R$ and $N = N_1 \overline{\otimes} R$ with M_1 and N_1 close.

5.1. **Solid and strongly solid factors.** As indicated to us by Kadison, one of the original motivations for the introduction of perturbation theory in [51] was to study the free group factors. Here we investigate the behavior of factors close to free group factors in the light of developments in the structure theory of these algebras. In [69], Ozawa discovered a remarkable property of certain II_1 factors, showing that the von Neumann algebra of a hyperbolic group is *solid* in the sense that every diffuse unital von Neumann subalgebra

$B \subseteq M$ has an amenable relative commutant $B' \cap M$. Subsequently Ozawa and Popa generalized the concept of solidity further: a II_1 factor M is said to be *strongly solid* if every unital diffuse amenable subalgebra $B \subseteq M$ has an amenable normalizing algebra $\mathcal{N}(B \subseteq M)''$. In [70], they extended Voiculescu's result that free group factors do not contain Cartan masas [103] by showing that free group factors are strongly solid. Subsequently a range of strongly solid factors have been discovered (see [71, 41, 14, 97] for example). Both of these properties are inherited by sufficiently close algebras, as we now show.

Proposition 5.1.1. *Let M and N be II_1 factors acting nondegenerately on a Hilbert space $\mathcal{H}$ with $d(M,N) < \gamma$.*

(i) *Suppose that $\gamma < 1/28669$. Then M is solid if and only if N is solid.*
(ii) *Suppose that $\gamma < 1/3200$. Then M is strongly solid if and only if N is strongly solid.*

Proof. (i). Suppose that M is solid and let $B_0 \subseteq N$ be diffuse. Choose a masa B in B_0 and note that B is diffuse since B_0 has this property. It suffices to show that $B' \cap N$ is amenable as then $B_0' \cap N \subseteq B' \cap N$ is also amenable since N is finite. Amenability of B allows us to apply Theorem 3.1.1 (i) to obtain a unitary $u \in (B \cup M)''$ with $uBu^* \subseteq M$, $\|u - I_{\mathcal{H}}\| \leq 150\gamma$ and $d(uBu^*, B) \leq 100\gamma$. Define $A = uBu^* \subseteq M$. Lemma 2.4.5 (i) shows that $A' \cap M \subseteq_\eta B' \cap N$ and $B' \cap N \subseteq_\eta A' \cap M$ where $\eta = 200\sqrt{2}\gamma + \gamma < 1/101$ by the choice of the bound on γ. Amenability of $A' \cap M$ follows from the assumption that M is solid, so Theorem 3.1.1 (i) gives an isomorphism between $A' \cap M$ and $B' \cap N$ so $B' \cap N$ is amenable. Thus N is solid. The reverse implication follows by interchanging the roles of M and N.

(ii). Without loss of generality, suppose that N is strongly solid and let P be a unital diffuse amenable von Neumann subalgebra of M. By Theorem 3.1.1 (i), there exists a unital von Neumann subalgebra $Q \subseteq N$ isomorphic to P such that $d(P,Q) \leq 100\gamma$. Now $\mathcal{N}(Q \subseteq N)''$ is amenable by assumption, and thus another application of Theorem 3.1.1 (i) gives a unitary $u \in (\mathcal{N}(Q \subseteq N) \cup M)''$ with $u\mathcal{N}(Q \subseteq N)''u^* \subseteq M$, $\|u - I_{\mathcal{H}}\| \leq 150\gamma$ and $d(u\mathcal{N}(Q \subseteq N)''u^*, \mathcal{N}(Q \subseteq N)'') \leq 100\gamma$. Then (2.2) gives $P \subseteq_{400\gamma} uQu^* \subseteq_{400\gamma} P$ and so, since our hypothesis ensures that $400\gamma < 1/8$, Theorem 3.1.1 (iii) gives a unitary $u_1 \in (P \cup uQu^*)'' \subseteq M$ with $\|u_1 - I_{\mathcal{H}}\| \leq 2800\gamma$ and $u_1 P u_1^* = uQu^*$.

Consider $P = u_1^* uQu^* u_1$ as a subalgebra of $N_1 := u_1^* uNu^* u_1$ and note that

$$\begin{aligned} d(M, N_1) &= d(u_1^* M u_1, uNu^*) = d(M, uNu^*) \\ &\leq d(M,N) + 2\|u - I_{\mathcal{H}}\| < 301\gamma. \end{aligned} \tag{5.1}$$

Now $u\mathcal{N}(Q \subseteq N)u^* \subseteq M$ and $u_1 \in M$ so $u_1^* u\mathcal{N}(Q \subseteq N)u^* u_1 \subseteq M$. We also have that $\mathcal{N}(P \subseteq N_1)'' = u_1^* u\mathcal{N}(Q \subseteq N)''u^* u_1$, and it follows that $\mathcal{N}(P \subseteq N_1)''$ is an amenable subalgebra of M. Moreover, $P' \cap N_1 \subseteq \mathcal{N}(P \subseteq N_1)'' \subseteq M$, implying that $P' \cap N_1 \subseteq P' \cap M$. Then Lemma 2.4.5 (i) (with $\delta = 0$) gives $P' \cap M \subseteq_{301\gamma} P' \cap N_1$ and so $P' \cap M = P' \cap N_1$ since $301\gamma < 1$ (this is a folklore Banach space argument, see [28, Proposition 2.4] for the precise statement being used). Let $\gamma_1 := 301\gamma$, yielding $M \subset_{\gamma_1} N_1 \subset_{\gamma_1} M$.

To establish that $\mathcal{N}(P \subseteq M)'' \subseteq \mathcal{N}(P \subseteq N_1)''$, consider a unitary $v \in \mathcal{N}(P \subseteq M)$. We apply Lemma 3.2.1 (iii) (with $\gamma_1 < 1/(2\sqrt{2})$ replacing γ) to obtain a unitary $v' \in \mathcal{N}(P \subseteq N_1)$ with $\|v - v'\| \leq (4+2\sqrt{2})\gamma_1$. Since $(4+2\sqrt{2})\gamma_1 = (4+2\sqrt{2})(301)\gamma < (4+2\sqrt{2})(301)/3200 < 1$, Proposition 2.6.2 gives unitaries $w \in P$ and $w' \in P'$ satisfying $v' = vww'$. Then $w' = w^* v^* v' \in P' \cap M$ since w, v, and v' all lie in M. Thus $w' \in P' \cap N_1 \subseteq \mathcal{N}(P \subseteq N_1)$. Then $v = v'w'^* w^* \in \mathcal{N}(P \subseteq N_1)$, proving that $\mathcal{N}(P \subseteq M) \subseteq \mathcal{N}(P \subseteq N_1)$. By assumption

$\mathcal{N}(P \subseteq N_1)''$ is amenable so the same is true for its subalgebra $\mathcal{N}(P \subseteq M)''$. Thus M is strongly solid. □

Remark 5.1.2. In particular, it follows that if M is a factor sufficiently close to a free group factor $L\mathbb{F}_r$, then M is strongly solid, we can find masas $A_1, \dots A_r$ in M close to the canonical generator masas in $L\mathbb{F}_r$, and these masas will have Pukánszky invariant $\{\infty\}$ (see Remark 4.2.6). Further, as the generator masas are maximal injective in $L\mathbb{F}_r$ ([80]), we can argue in a similar fashion to the previous theorem to deduce that each A_i is maximal injective in M.

5.2. **Property Gamma and McDuff factors.** In this section we examine factors close to those with Property Gamma and factors close to McDuff factors. We begin by recording an extension of Lemma 2.4.4.

Lemma 5.2.1. *Let M and N be* II_1 *factors represented nondegenerately on a Hilbert space $\mathcal{H}$ and let γ and η be positive constants. Suppose that $d(M,N) < \gamma < 1$ and that we have x_1, x_2 in the unit ball of M and y_1, y_2 in the unit ball of N with $\|x_i - y_i\| \leq \eta$, $i = 1, 2$. Then*

$$\|y_1 - y_2\|_{2,N}^2 \leq \|x_1 - x_2\|_{2,M}^2 + 8\eta + (8\sqrt{2} + 8)\gamma. \tag{5.2}$$

Proof. Define $s = y_1 - y_2 \in N$ and $t = x_1 - x_2 \in M$, so that $\|s\|$, $\|t\| \leq 2$ and $\|s - t\| \leq 2\eta$. Let Φ be a state on $\mathcal{B}(\mathcal{H})$ extending τ_M. Then Lemma 2.4.4 gives

$$|\tau_N(s^*s) - \Phi(s^*s)| \leq (2\sqrt{2} + 2)\gamma\|s^*s\| \leq (8\sqrt{2} + 8)\gamma. \tag{5.3}$$

We also have

$$|\Phi(s^*s) - \Phi(t^*t)| \leq \|(s^* - t^*)s + t^*(s - t)\| \leq 8\eta, \tag{5.4}$$

so

$$\begin{aligned} \|s\|_{2,N}^2 = \tau_N(s^*s) &\leq |\Phi(s^*s)| + |\tau_N(s^*s) - \Phi(s^*s)| \\ &\leq |\Phi(s^*s) - \Phi(t^*t)| + \Phi(t^*t) + (8\sqrt{2} + 8)\gamma \\ &\leq \|t\|_{2,M}^2 + 8\eta + (8\sqrt{2} + 8)\gamma, \end{aligned} \tag{5.5}$$

since Φ and τ_M agree on M. This is (5.2). □

For von Neumann algebras M with separable predual, the definition of property Gamma is equivalent to the condition $M' \cap M^\omega \neq \mathbb{C}I$ for a nonprincipal ultrafilter ω on $\mathbb{N}$ (see [101, Theorem XIV.4.7]). For II_1 factors with nonseparable preduals this equivalence no longer holds (see [36, Section 3]) and instead one must work with ultrafilters on sets of larger cardinality. For simplicity, we restrict to the separable predual situation below. However the argument can be modified to handle the nonseparable situation (with the same constants).

Proposition 5.2.2. *Let M and N be* II_1 *factors with separable preduals acting nondegenerately on a Hilbert space $\mathcal{H}$ with $d(M,N) < \gamma$ for a constant $\gamma < 1/190$. Suppose that M has property Γ. Then N also has property Γ.*

Proof. Suppose that M has property Γ and fix a nonprincipal ultrafilter ω on $\mathbb{N}$. Murray and von Neumann's definition of property Γ [66] gives a sequence $(u_n)_{n=1}^\infty$ of trace zero unitaries such that $u = (u_n) \in M^\omega \cap M'$. For each n, use Lemma 2.4.1 (i) to find a unitary $v_n \in N$ with $\|u_n - v_n\| < \sqrt{2}\gamma$ and let v denote the class of (v_n) in N^ω. Letting Φ denote a state on $\mathbb{B}(\mathcal{H})$ extending τ_N, Lemma 2.4.4 gives the estimate

$$|\tau_M(u_n) - \Phi(u_n)| \leq (2\sqrt{2} + 2)\gamma, \quad n \in \mathbb{N}, \tag{5.6}$$

so that

$$|\tau_M(u_n)-\tau_N(v_n)|\leq|\tau_M(u_n)-\Phi(u_n)|+|\Phi(u_n)-\Phi(v_n)|\leq(3\sqrt{2}+2)\gamma. \tag{5.7}$$

Thus

$$|\tau_{N^\omega}(v)|\leq(3\sqrt{2}+2)\gamma. \tag{5.8}$$

Given a unitary $w\in N$, use Lemma 2.4.1 (i) to find a unitary $w'\in M$ with $\|w'-w\|<\sqrt{2}\gamma$. Then

$$\|w'u_n-wv_n\|\leq\|(w'-w)u_n\|+\|w(u_n-v_n)\|\leq 2\sqrt{2}\gamma \tag{5.9}$$

and similarly $\|u_nw'-v_nw\|\leq 2\sqrt{2}\gamma$. Taking $\eta=2\sqrt{2}\gamma$ in Lemma 5.2.1 with $x_1=w'u_n$, $x_2=u_nw'$, $y_1=wv_n$ and $y_2=v_nw$ gives

$$\|wv_n-v_nw\|_{2,N}^2\leq\|w'u_n-u_nw'\|_{2,M}^2+(24\sqrt{2}+8)\gamma. \tag{5.10}$$

Since $\lim_{n\to\omega}\|w'u_n-u_nw'\|_{2,M}=0$ in N^ω, we have

$$\|wvw^*-v\|_{2,N^\omega}^2=\|wv-vw\|_{2,N^\omega}^2\leq(24\sqrt{2}+8)\gamma. \tag{5.11}$$

Let y be the unique element of minimal $\|\cdot\|_{2,N^\omega}$-norm in $\overline{\mathrm{conv}}^{2,N^\omega}\{wvw^*:w\in\mathcal{U}(N)\}$. This lies in N^ω and uniqueness ensures that $y\in N^\omega\cap N'$. It remains to check that y is nontrivial.

The estimate (5.11) gives

$$\|y-v\|_{2,N^\omega}^2\leq(24\sqrt{2}+8)\gamma, \tag{5.12}$$

and so

$$\|y\|_{2,N^\omega}\geq 1-((24\sqrt{2}+8)\gamma)^{1/2} \tag{5.13}$$

as $\|v\|_{2,N^\omega}=1$. We can estimate

$$\begin{aligned}|\tau_{N^\omega}(y)|&\leq|\tau_{N^\omega}(v)|+|\tau_{N^\omega}(y-v)|\\&\leq(3\sqrt{2}+2)\gamma+\|y-v\|_{2,N^\omega}\\&\leq(3\sqrt{2}+2)\gamma+((24\sqrt{2}+8)\gamma)^{1/2},\end{aligned} \tag{5.14}$$

using (5.8), (5.13) and the Cauchy-Schwarz inequality. If $y\in\mathbb{C}I_{N^\omega}$, then $y=\tau_{N^\omega}(y)I_{N^\omega}$ so

$$1-((24\sqrt{2}+8)\gamma)^{1/2}\leq\|y\|_{2,N^\omega}\leq(3\sqrt{2}+2)\gamma+((24\sqrt{2}+8)\gamma)^{1/2}. \tag{5.15}$$

Direct computations show that this is a contradiction when $\gamma<1/190$, so that y is a nontrivial element of $N'\cap N^\omega$. Therefore N has property Gamma. □

Corollary 5.2.3. *Let M be a weakly Kadison-Kastler stable II_1 factor with property Gamma and separable predual. Then M is Kadison-Kastler stable.*

Proof. Suppose that $M\subseteq\mathcal{B}(\mathcal{H})$ is a nondegenerate normal representation and that $N\subseteq\mathcal{B}(\mathcal{H})$ has $d(M,N)$ small enough so that $M\cong N$. Then N also has property Gamma, and so Proposition 2.2.4 (ii) and Properties 2.2.2 (i) give $d_{cb}(M,N)\leq 10d(M,N)$. If additionally $d(M,N)<1/(10\times 301\times 136209)$, then Proposition 4.3.1 gives $\dim_M(\mathcal{H})=\dim_N(\mathcal{H})$ and so an isomorphism between M and N is spatially implemented on $\mathcal{H}$ [72] (see also [33, Section 6.4, Proposition 10]). □

We now turn to McDuff factors, defined as those II_1 factors M with $M \cong M \overline{\otimes} R$, where R is the hyperfinite II_1 factor. In the separable predual case, one can use McDuff's characterization of these factors as those M for which $M' \cap M^\omega$ is nonabelian [58], and transfer a suitably noncommuting pair of centralizing sequences from a McDuff factor M into a nearby factor N to see that N is also McDuff. The details are similar in style to Proposition 5.2.2. However, by using the reduction techniques of Section 4, we can go further and show that if M absorbs R tensorially, then so too does N, and that we can arrange to simultaneously pull out the same tensor factor of R from M and N (after making a small unitary perturbation). Recall from [67] that if N is a II_1 factor and Q_1 and Q_2 are commuting subfactors of N, then N is generated by this pair if and only if $N \cong Q_1 \overline{\otimes} Q_2$.

Lemma 5.2.4. *Let M be a McDuff factor with a separable predual acting nondegenerately on a Hilbert space $\mathcal{H}$ and write $M = M_0 \overline{\otimes} R$ for some II_1 factor M_0 on $\mathcal{H}$. Suppose that N is another II_1 factor on $\mathcal{H}$ with $d(M,N) < \gamma < 1/(305 \times 903 \times 47)$. Given an amenable subalgebra P_0 of M_0 with $P_0' \cap M_0 \subseteq P_0$, there exists a unitary $u \in (M \cup N)''$ with $\|u - I_{\mathcal{H}}\| \leq 150\gamma$ such that:*

(i) writing $R_1 = uRu^*$, we have $R_1 \subseteq N$;
(ii) $N \cong (R_1' \cap N) \overline{\otimes} R_1$;
(iii) $R_1' \cap N \subseteq_{cb,(200\sqrt{2}+5)\gamma} M_0$ and $M_0 \subseteq_{cb,(200\sqrt{2}+5)\gamma} R_1' \cap N$;
(iv) $uP_0u^* \subseteq R_1' \cap N$.

In particular N is McDuff.

Proof. As M is McDuff it certainly has property Gamma and hence so too does N by Lemma 5.2.2. Thus $M \subset_{cb,5\gamma} N$ and $N \subset_{cb,5\gamma} M$ by Proposition 2.2.4 (ii). Set $P = (P_0 \cup R)''$ so that $P' \cap M \subseteq P$. By Theorem 3.1.1 (i), there exists a unitary $u \in (P \cup N)''$ with $\|u - I_{\mathcal{H}}\| \leq 150\gamma$, $uPu^* \subseteq N$, and $\|uxu^* - x\| \leq 100\gamma\|x\|$ for $x \in P$. Set $N_1 = u^*Nu$ so that $M \subseteq_{cb,305\gamma} N_1$, $N_1 \subseteq_{cb,305\gamma} M$ and $P \subseteq N_1$.

As $uRu^* \subseteq_{100\gamma} R$, Lemma 2.4.5 (ii) gives $R' \cap M \subseteq_{cb,200\sqrt{2}\gamma+5\gamma} (uRu^*)' \cap N$. Similarly, $(uRu^*)' \cap N \subseteq_{cb,200\sqrt{2}\gamma+5\gamma} R' \cap M$. In particular $(uRu^*)' \cap N$ is a factor by [51, Corollary A], and hence so too is $R' \cap N_1$. It remains to show that R and $R' \cap N_1$ generate N_1, as then $N_1 \cong (R' \cap N_1) \overline{\otimes} R$ by [67].

By applying Theorem 4.3.3 (valid as $305\gamma < 1/(903 \times 47)$) we can find unital normal representations $\pi : M \to \mathcal{B}(\mathcal{K})$ and $\rho : N_1 \to \mathcal{B}(\mathcal{K})$ such that $\pi|_P = \rho|_P$ and, writing $M_2 = \pi(M)$ and $N_2 = \rho(N_1)$, we also obtain that the algebras M_2, M_2', N_2 and N_2' have common trace vector ξ on $\mathcal{K}$. Moreover, $M_2 \subset_{cb,\beta} N_2$ and $N_2 \subset_{cb,\beta} M_2$ for $\beta = 903 \times 305\gamma$, while $J_{M_2}\pi(P)J_{M_2} = J_{N_2}\pi(P)J_{N_2} \subseteq N_2'$. Write $R_2 = \rho(R) \subseteq M_2 \cap N_2$. Now fix $x \in R_2' \cap M_2$ and $\varepsilon > 0$. Choose $y \in N_2$ such that $\|y\xi - x\xi\|_{\mathcal{K}} \leq \varepsilon$. For each unitary $u \in R_2$, x commutes with u and $J_{N_2}uJ_{N_2} \in N_2'$. Therefore $uJ_{N_2}uJ_{N_2}x\xi = xuJ_{N_2}uJ_{N_2}\xi = x\xi$. Thus

$$\|x\xi - uyu^*\xi\|_{\mathcal{K}} = \|uJ_{N_2}uJ_{N_2}(x\xi - y\xi)\|_{\mathcal{K}} \leq \varepsilon. \tag{5.16}$$

Now $E^{N_2}_{R_2' \cap N_2}(y)\xi$ is the vector in $\mathcal{K}$ of minimal norm in the closed convex hull of $\{uyu^*\xi : u \in \mathcal{U}(R_2)\}$, so

$$\|x\xi - E^{N_2}_{R_2' \cap N_2}(y)\xi\|_{\mathcal{K}} \leq \varepsilon. \tag{5.17}$$

For any unitary $v \in R_2$, we have $\|vx\xi - vE^{N_2}_{R_2'\cap N_2}(y)\xi\|_{\mathcal{K}} \leq \varepsilon$. Since ε was arbitrary we conclude that

$$\mathcal{K} = \overline{((R_2'\cap M_2)\cup R_2)''\xi} = \overline{\mathrm{Span}\{vx\xi : v \in \mathcal{U}(R_2),\ x \in R_2'\cap M_2\}}$$
$$\subseteq \overline{((R_2'\cap N_2)\cup R_2)''\xi}. \tag{5.18}$$

Since ξ is separating for N_2, it follows that $N_2 = ((R_2'\cap N_2)\cup R_2)''$ just as in the proof of Lemma 4.1.4 (iii). Thus N_1 is generated by R and $R'\cap N_1$. $\square$

Proposition 5.2.5. *Suppose that M is a weakly Kadison-Kastler stable II_1 factor. Then $M\,\overline{\otimes}\,R$ is Kadison-Kastler stable, where R is the hyperfinite II_1 factor.*

Proof. Let $\delta > 0$ be so small that, given a unital normal representation $M \subseteq \mathcal{B}(\mathcal{H})$, each II_1 factor $N_0 \subseteq \mathcal{B}(\mathcal{H})$ with $d(M, N_0) < \delta$ is isomorphic to M. Now suppose that $M\,\overline{\otimes}\,R$ and N are represented on $\mathcal{K}$ with $d(M\,\overline{\otimes}\,R, N) < \gamma$ for some $\gamma < 1/(305 \times 903 \times 47)$. Lemma 5.2.4 shows that $N \cong N_0\,\overline{\otimes}\,R_1$ for another copy of the hyperfinite II_1 factor R_1 and a II_1 factor N_0 with $N_0 \subseteq_{cb,(200\sqrt{2}+5)\gamma} M_0$ and $M_0 \subseteq_{cb,(200\sqrt{2}+5)\gamma} N_0$. Thus, provided $2((200\sqrt{2}+5)\gamma) < \delta$, M_0 and N_0 are isomorphic, whence M and N are isomorphic. This shows that $M\,\overline{\otimes}\,R$ is weakly Kadison Kastler stable. As $M\,\overline{\otimes}\,R$ has property Gamma, it is then Kadison-Kastler stable by Corollary 5.2.3. $\square$

Note that if M satisfies the conclusions of Part (3) of Theorem B, then strong Kadison-Kastler stability of $M\,\overline{\otimes}\,R$ can also be established, see Proposition 6.3.7.

6. Kadison-Kastler stable factors

We now present the main results of the paper by exhibiting classes of actions giving rise to Kadison-Kastler stable crossed product factors. In Section 6.1 we first show that a von Neumann algebra close to a twisted crossed product II_1 factor $P \rtimes_{\alpha,\omega} \Gamma$ where P is amenable must also be a twisted crossed product factor arising from the same action and some cocycle ω' which is uniformly close to ω. From this we are able to prove Part (1) of Theorem B (Corollary 6.1.2), which in particular shows that crossed products $L^\infty(X) \rtimes \mathbb{F}_r$ are weakly Kadison-Kastler stable (Corollary 6.1.3). In Section 6.2 we use property Gamma to obtain Kadison-Kastler stable factors, those that are spatially isomorphic to sufficiently close neighbors. Corollary 6.2.1 proves Part (2) of Theorem B and we also discuss how to produce examples of actions satisfying the hypothesis of this result. In Section 6.3 we turn to strong Kadison-Kastler stability and prove Theorem A as Theorem 6.3.4 and Part (3) of Theorem B as Theorem 6.3.2. Remark 6.3.6 sets out examples of factors to which these results apply.

We then discuss what our methods give in more general situations arising from regular subalgebras which are not twisted crossed products. Subsection 6.4 examines inclusions $A \subseteq M$ of Cartan masas in II_1 factors. Analogously to the twisted crossed product situation, we show that if N is close to M, then every Cartan masa B in N close to A (these must exist by the results of Sections 4.1 and 4.2) induces the same measured equivalence relation as the original inclusion $A \subseteq M$ and, further, M and N arise from uniformly close cocycles on this relation. Similar results hold when M contains a regular amenable irreducible subfactor.

6.1. **Weakly Kadison-Kastler stable crossed products.** The work of Section 4 enables us to transfer a twisted crossed product structure from a II_1 factor to nearby factors. Recall (see Section 2.5) that every 2-cocycle is cohomologous to a normalized 2-cocycle so there is no loss of generality in only considering normalized cocycles below.

Theorem 6.1.1. *Let $\alpha : \Gamma \curvearrowright P$ be a trace preserving, centrally ergodic, and properly outer action of a countable discrete group Γ on a finite amenable von Neumann algebra P with separable predual. Let $\omega \in Z^2(\Gamma, \mathcal{U}(\mathcal{Z}(P)))$ be a normalized 2-cocyle, write $M = P \rtimes_{\alpha,\omega} \Gamma$ and suppose that $M \subseteq \mathcal{B}(\mathcal{H})$ is a unital normal representation. Let N be another von Neumann algebra on $\mathcal{H}$ with $d(M,N) < \gamma$ for some $\gamma < 5.77 \times 10^{-16}$. Then there exists a normalized 2-cocycle $\omega' \in Z^2(\Gamma, \mathcal{U}(\mathcal{Z}(P)))$ such that $N \cong P \rtimes_{\alpha,\omega'} \Gamma$ and*

$$\sup_{g,h\in\Gamma} \|\omega(g,h) - \omega'(g,h)\| < 14889\gamma < 8.6 \times 10^{-12}. \tag{6.1}$$

Proof. By Proposition 2.5.3, M is a II_1 factor and $P' \cap M \subseteq P$. The bound on γ ensures that Kadison and Kastler's stability of type classification from [51] applies and so N is also a II_1 factor. Since $\gamma < 1/100$, Theorem 3.1.1 (i) provides a unitary $u \in (P \cup N)''$ with $\|I_{\mathcal{H}} - u\| \leq 150\gamma$ such that $P \subseteq N_1 := u^*Nu$. Moreover, we have $M \subset_{\gamma_1} N_1 \subset_{\gamma_1} M$, where $\gamma_1 = 301\gamma$. Write $(u_g)_{g\in\Gamma}$ for the canonical bounded homogeneous orthonormal basis of normalizers for $P \subseteq M$ implementing the action α which satisfy $u_g u_h = \omega(g,h)u_{gh}$ for $g, h \in \Gamma$. For $g \in \Gamma$, we can apply Lemma 3.2.1 (iii) to M and N_1 to obtain a normalizer $w_g \in \mathcal{N}(P \subseteq N_1)$ with

$$\|w_g - u_g\| \leq (4 + 2\sqrt{2})\gamma_1 < 1. \tag{6.2}$$

Thus, by Proposition 2.6.2, $w_g = u_g p_g p_g'$ for some unitaries $p_g \in P$ and $p_g' \in P'$ with $\|p_g - I_{\mathcal{H}}\| \leq 2^{1/2}\|w_g - u_g\| \leq 2^{1/2}(4 + 2\sqrt{2})\gamma_1$. Write $v_g = w_g p_g^* = u_g p_g' \in N_1$ so that $v_g x v_g^* = u_g x u_g^* = \alpha_g(x)$ for all $x \in P$ and

$$\|v_g - u_g\| \leq (1 + \sqrt{2})(4 + 2\sqrt{2})\gamma_1 = (8 + 6\sqrt{2})\gamma_1 < 4963\gamma. \tag{6.3}$$

Since $u_e = I_{\mathcal{H}}$, we may assume that $v_e = w_e = I_{\mathcal{H}}$. For use in Lemma 6.3.1, note that if N happens to already contain P then we can take $N = N_1$ and $u = I_{\mathcal{H}}$, and the estimate in (6.3) is replaced by

$$\|v_g - u_g\| \leq (8 + 6\sqrt{2})\gamma < 16.5\gamma. \tag{6.4}$$

The bound on γ in the statement of the theorem is chosen so that $\gamma_1 < 1.74 \times 10^{-13}$ and so Lemma 4.2.3 applies as $P' \cap M \subseteq P$. In particular $P' \cap N_1 \subseteq P$, while Lemma 4.2.3 (viii) and the estimate (6.3) show that $(w_g)_{g\in\Gamma}$ is a bounded homogeneous orthonormal basis of normalizers for $P \subseteq N_1$. Then $(v_g)_{g\in\Gamma}$ also has this property and so N_1 is generated by P and the normalizers $(v_g)_{g\in\Gamma}$. As we have $E_P^{N_1}(v_g) = 0$ for $g \in \Gamma \setminus \{e\}$ and $v_g x v_g^* = \alpha_g(x)$ for $x \in P$, Proposition 2.5.1 shows that N_1 is isomorphic to the twisted crossed product $P \rtimes_{\alpha,\omega'} \Gamma$, where ω' is given by

$$\omega'(g,h) = v_g v_h v_{gh}^* \in \mathcal{U}(\mathcal{Z}(P)), \quad g, h \in \Gamma. \tag{6.5}$$

Since we chose $v_e = I_{\mathcal{H}}$, this 2-cocycle is normalized and applying the estimate (6.3) three times gives

$$\begin{aligned}\|\omega(g,h) - \omega'(g,h)\| &= \|u_g u_h u_{gh}^* - v_g v_h v_{gh}^*\| \\ &\leq \|u_g - v_g\| + \|u_h - v_h\| + \|u_{gh}^* - v_{gh}^*\| \\ &< 3 \times 4963\gamma = 14889\gamma \\ &< 8.6 \times 10^{-12}, \quad g,h \in \Gamma. \end{aligned} \tag{6.6}$$

Since N and N_1 are unitarily conjugate, $N \cong P \rtimes_{\alpha,\omega'} \Gamma$ for this 2-cocycle. □

We obtain weakly Kadison-Kastler stable factors whenever we can guarantee that the uniformly close cocycles ω and ω' in Theorem 6.1.1 are cohomologous. This happens when the comparison map from bounded to usual cohomology vanishes in degree 2. In particular, the next corollary proves Part (1) of Theorem B.

Corollary 6.1.2. *Let $\alpha : \Gamma \curvearrowright P$ be a trace preserving, centrally ergodic, and properly outer action of a countable discrete group Γ on a finite amenable von Neumann algebra P with separable predual. Suppose that one of the comparison maps*

$$H_b^2(\Gamma, \mathcal{Z}(P)_{sa}) \to H^2(\Gamma, \mathcal{Z}(P)_{sa}) \tag{6.7}$$

or

$$H_b^2(\Gamma, L^2(\mathcal{Z}(P)_{sa})) \to H^2(\Gamma, L^2(\mathcal{Z}(P)_{sa})) \tag{6.8}$$

vanishes. For any $\omega \in H^2(\Gamma, \mathcal{U}(\mathcal{Z}(P)))$, let M be the crossed product II_1 factor $P \rtimes_{\alpha,\omega} \Gamma$, faithfully, normally, and nondegenerately represented on a Hilbert space $\mathcal{H}$. Then M has the property that $M \cong N$ whenever N is a von Neumann algebra on $\mathcal{H}$ satisfying $d(M,N) < 5.77 \times 10^{-16}$, and thus M is weakly Kadison-Kastler stable.

Proof. Let $M := P \rtimes_{\alpha,\omega} \Gamma$ for a normalized 2-cocycle ω and suppose that M is faithfully normally and nondegenerately represented on $\mathcal{H}$. Given another von Neumann algebra N on $\mathcal{H}$ with $d(M,N) < 5.77 \times 10^{-16}$, we have $N \cong P \rtimes_{\alpha,\omega'} \Gamma$ for some normalized $\omega' \in Z^2(\Gamma, \mathcal{U}(\mathcal{Z}(P)))$ satisfying (6.1) by Theorem 6.1.1. Define a 2-cocycle $\nu \in Z^2(\Gamma, \mathcal{U}(\mathcal{Z}(P)))$ by $\nu(g,h) = \omega(g,h)\omega'(g,h)^*$ so that

$$\sup_{g,h\in\Gamma} \|\nu(g,h) - I_P\| < 8.6 \times 10^{-12} < \sqrt{2}. \tag{6.9}$$

By Lemma 2.5.2 (i), $\psi := -i \log \nu$ defines a bounded 2-cocycle in $Z_b^2(\Gamma, \mathcal{Z}(P)_{sa})$. If we assume that the comparison map (6.7) vanishes, then $\psi = \partial\phi$ for some $\phi \in C^1(\Gamma, \mathcal{Z}(P)_{sa})$. In this case Lemma 2.5.2 (ii) shows that $\nu = \partial e^{i\phi}$ so that ω and ω' represent the same element of $H^2(\Gamma, \mathcal{U}(\mathcal{Z}(P))$. By Proposition 2.5.1 (ii), M and N are isomorphic.

When the comparison map (6.8) vanishes, then we regard ψ as taking values in $L^2(\mathcal{Z}(P)_{sa})$ and use Lemma 2.5.2 (iii) to see that ν is trivial in $H^2(\Gamma, \mathcal{U}(\mathcal{Z}(P)))$. Again M and N are isomorphic by Proposition 2.5.1 (ii). □

Free groups have cohomological dimension one, so $H^2(\mathbb{F}_r, \mathcal{Z}(P)_{sa}) = 0$ for all actions $\alpha : \mathbb{F}_r \curvearrowright P$. Consequently, the comparison map (6.7) is zero and the resulting crossed products are weakly Kadison-Kastler stable. In particular all crossed products $L^\infty(X) \rtimes_\alpha \mathbb{F}_r$ for free, ergodic, and probability measure preserving actions are weakly Kadison-Kastler stable.

Corollary 6.1.3. *Let $\mathbb{F}_r$ be a free group of rank $r = 2, 3, \ldots, \infty$ and let $\alpha : \mathbb{F}_r \curvearrowright P$ be a trace preserving, centrally ergodic, properly outer action on a finite amenable von Neumann algebra P with separable predual. Then $P \rtimes_\alpha \mathbb{F}_r$ is weakly Kadison-Kastler stable.*

The following corollary is the wreath product version of Kadison's question about algebras close to free group factors. Note that for each r, Bowen has shown that $L(\Gamma_1 \wr \mathbb{F}_r) \cong L(\Gamma_2 \wr \mathbb{F}_r)$ for nontrivial abelian groups Γ_1 and Γ_2, [5] (see [102]).

Corollary 6.1.4. *Let Γ be a nontrivial abelian group and let $r = 2, 3, \ldots, \infty$. Then $L(\Gamma \wr \mathbb{F}_r)$ is weakly Kadison-Kastler stable.*

Proof. Since $L(\Gamma \wr \mathbb{F}_r)$ is a crossed product $L(\Gamma^{\mathbb{F}_r}) \rtimes \mathbb{F}_r$ arising from a free, ergodic, trace preserving action, the result follows from Corollary 6.1.3. □

Corollary 6.1.2 also applies when the bounded cohomology group $H_b^2(\Gamma, \mathcal{Z}(P)_{sa}) = 0$ or when $H_b^2(\Gamma, L^2(\mathcal{Z}(P)_{sa})) = 0$, for example when $\Gamma = SL_n(\mathbb{Z})$ for $n \geq 3$. However for these groups we can use the vanishing of $H_b^2(\Gamma, \mathcal{Z}(P)_{sa}) = 0$ to extract more information, see Section 6.3.

6.2. **Kadison-Kastler stable factors.** We obtain Kadison-Kastler stable factors from the examples in Section 6.1 by imposing additional conditions which ensure that the isomorphism in Corollary 6.1.2 is spatially implemented. In particular, as factors with property Gamma have the similarity property, one can convert weak Kadison-Kastler stability to Kadison-Kastler stability. The corollary below proves Part (2) of Theorem B.

Corollary 6.2.1. *Let $\alpha : \Gamma \curvearrowright P$ be a properly outer, centrally ergodic, trace preserving action of a countable discrete group on a finite amenable von Neumann algebra P with separable predual and suppose that one of the comparison maps $H_b^2(\Gamma, \mathcal{Z}(P)_{sa}) \to H^2(\Gamma, \mathcal{Z}(P)_{sa})$ or $H_b^2(\Gamma, L^\infty(\mathcal{Z}(P)_{sa})) \to H^2(\Gamma, L^2(\mathcal{Z}(P)_{sa}))$ vanishes. If the crossed product factor $P \rtimes_\alpha \Gamma$ has property Gamma, then it is Kadison-Kastler stable.*

Proof. Since the crossed product $P \rtimes_\alpha \Gamma$ has property Gamma, this follows by combining Corollary 6.1.2, which shows that such a crossed product is weakly Kadison-Kastler stable, and Corollary 5.2.3. □

Remark 6.2.2. One can obtain explicit constants in the previous corollary, which are better than those appearing in Corollary 6.1.2 by arguing as follows. When $M = P \rtimes_\alpha \Gamma$ and N is an algebra sufficiently close to M on the Hilbert space $\mathcal{H}$, note that N also has property Gamma by Proposition 5.2.2. Consequently M and N are completely close by Proposition 2.2.4. We can then use Theorem 4.3.3 in place of Lemma 4.2.3 in the proof of Theorem 6.1.1 to obtain that M and N are isomorphic. Since Proposition 4.3.1 shows that $\dim_{\mathcal{H}} M = \dim_{\mathcal{H}} N$, this isomorphism is automatically spatially implemented on $\mathcal{H}$. The hypotheses of all the results quoted in this brief sketch are met when $d(M, N) < (301 \times 136209)^{-1}$.

We now turn to examples where Corollary 6.2.1 applies. Consider a II_1 factor M arising as a crossed product $M = P \rtimes_\alpha \Gamma$ for some trace preserving action of a countable discrete group on a finite von Neumann algebra P. Assume further that the group Γ is not inner amenable in the sense of [34] (such as a free group of rank at least 2 [34] or an ICC group with property T [1, Theorem 11]). Under this hypothesis, any central sequence $(x_n)_{n=1}^\infty$ in M is asymptotically contained in P (this assertion is proved in [40, Lemma 1] when Γ is a free group and the method of proof works whenever Γ is not inner amenable). Consequently we must look for examples where centralizing sequence for M can be found in P. First we consider the case when P is abelian.

Given an ergodic measure preserving action $\alpha : \Gamma \curvearrowright (X, \mu)$, there are nontrivial centralizing sequences for the crossed product factor $L^\infty(X) \rtimes_\alpha \Gamma$ inside $L^\infty(X)$ if and only if the action fails to be strongly ergodic [30, 92]. This means that there is a sequence $(Y_n)_{n=1}^\infty$ of Borel subsets of X satisfying the asymptotic invariance condition $\mu(Y_n \Delta (g \cdot Y_n)) \to 0$ for all $g \in \Gamma$ which are nontrivial in the sense that $\liminf_n \mu(Y_n)(1 - \mu(Y_n)) > 0$. When Γ has Každan's property T every ergodic action is strongly ergodic [91], and so there are no actions of $SL_n(\mathbb{Z})$ for $n \geq 3$ to which the previous corollary applies. Conversely, Connes and Weiss show that if Γ does not have property T, then Γ admits ergodic (and in fact weakly mixing) actions which are not strongly ergodic ([30]). Further, these actions can be taken to be essentially free (see [13, Proposition 2.2.3]). In particular, the free groups $\mathbb{F}_r$ admit free, ergodic measure preserving actions which fail to be strongly ergodic and so Corollary 6.2.1 applies to the crossed products arising from these actions. As noted in [70, Section 5] one can construct such actions of free groups which are additionally profinite.

Corollary 6.2.3. *Let $\alpha : \mathbb{F}_r \curvearrowright (X, \mu)$ be a free, ergodic, probability measure preserving action which is not strongly ergodic. Then $L^\infty(X, \mu) \rtimes_\alpha \Gamma$ is Kadison-Kastler stable.*

Remark 6.2.4. Corollary 6.2.1 can also be used for actions on nonabelian von Neumann algebras. Just as in the abelian situation, when Γ does not have property T there exist properly outer actions $\alpha : \Gamma \curvearrowright R$ on the hyperfinite II_1 factor for which the resulting crossed product has property Gamma, as we now explain. An outer action $\beta : \Gamma \curvearrowright R$ is said to be strong if every bounded asymptotically invariant sequence $(x_n)_{n=1}^\infty$ (as defined by the requirement that $\|\beta_g(x_n) - x_n\|_2 \to 0$ for all $g \in \Gamma$) is equivalent to a sequence in the fixed point algebra R^Γ. Recall that if Γ does not have property T, then there exists an outer action $\beta : \Gamma \curvearrowright R$ that is not strong ([2], noting that the construction in (ii) on p. 208 is outer). Thus there exists a bounded asymptotically invariant sequence $(x_n)_{n=1}^\infty$ that does not represent a scalar multiple of the identity in R^ω. Now consider the infinite tensor product $R^{\otimes\infty}$ and the action $\alpha : \Gamma \curvearrowright R^{\otimes\infty}$ given by $\alpha_g(\bigotimes_m a_m) = \bigotimes_m \beta_g(a_m)$. This is easily seen to be an outer action. Defining $y_n = I_R^{\otimes(n-1)} \otimes x_n \otimes I_R \otimes \cdots$, we obtain a nontrivial centralizing sequence $(y_n)_n$ in $R^{\otimes\infty}$ for $R^{\otimes\infty} \rtimes_\alpha \Gamma$ and so the crossed product factor has property Gamma.

We end this section by noting that the tensor product of any of the weakly Kadison-Kastler stable factors from Section 6.1 with the hyperfinite II_1 factor is automatically Kadison-Kastler stable.

Corollary 6.2.5. *Let $\alpha : \Gamma \curvearrowright P$ be a trace preserving, centrally ergodic, and properly outer action of a countable discrete group Γ on a finite amenable von Neumann algebra P with separable predual. Suppose that one of the comparison maps (6.7) or (6.8) vanishes (as happens when Γ is a free group). Write $M = P \rtimes_\alpha \Gamma$. Then the II_1 factor $M \overline{\otimes} R$ is Kadison-Kastler stable, where R is the hyperfinite II_1 factor.*

Proof. The factors M are weakly Kadison-Kastler stable by Corollary 6.1.2, so $M \overline{\otimes} R$ is Kadison Kastler stable by Lemma 5.2.5. ∎

6.3. **Strongly Kadison-Kastler stable crossed products.** We now turn to the situation where the bounded cohomology groups $H_b^2(\Gamma, \mathcal{Z}(P)_{sa})$ vanish. In this case the isomorphisms we obtain between a crossed product factor and a nearby factor are uniformly close to the inclusion map (see Theorem 6.3.2 below). This enables us to produce strongly Kadison-Kastler stable factors in Theorem 6.3.4 below which proves Theorem A. We first examine the situation when the crossed product factor lies in standard position.

Lemma 6.3.1. *Let $\alpha : \Gamma \curvearrowright P$ be a trace preserving, centrally ergodic and properly outer action of a countable discrete group Γ on a finite amenable von Neumann algebra P with separable predual. Suppose that $H_b^2(\Gamma, \mathcal{Z}(P)_{sa}) = 0$. Given a normalized 2-cocycle $\omega \in Z^2(\Gamma, \mathcal{U}(\mathcal{Z}(P)))$, write $M = P \rtimes_{\alpha,\omega} \Gamma$ and suppose that $M \subseteq \mathcal{B}(\mathcal{K})$ is represented in standard position with tracial vector ξ used to define the modular conjugation operator J_M and the orthogonal projection e_P onto $\overline{P\xi}$. Let $N \subseteq \mathcal{B}(\mathcal{K})$ be another von Neumann algebra with $M \subseteq_\beta N$ and $N \subseteq_\beta M$ for $\beta < 1/47$ and such that $P \subseteq N$ and $J_M P J_M \subseteq N'$. Then there exists a unitary $U \in P' \cap \langle M, e_P \rangle$ such that $UMU^* = N$, and*

$$\|U - I_{\mathcal{K}}\| \leq (8 + 6\sqrt{2})\beta + 9K(3 + 2\sqrt{2})\beta \tag{6.10}$$

where K is the constant (depending only on the action α) given by the open mapping theorem such that (2.29) holds.

Proof. Write $(u_g)_{g\in\Gamma}$ for the canonical unitaries in $M = P \rtimes_{\alpha,\omega} \Gamma$ satisfying $u_g u_h = \omega(g,h)u_{gh}$ for $g, h \in \Gamma$. Just as in the proof of Theorem 6.1.1 (see equation (6.4)), we can find unitaries $(v_g)_{g\in\Gamma}$ in N satisfying

$$\|v_g - u_g\| \leq (8 + 6\sqrt{2})\beta \tag{6.11}$$

such that $v_e = u_e = I_{\mathcal{K}}$, $v_g x v_g^* = u_g x u_g^* = \alpha_g(x)$ for $x \in P$ and $(v_g)_{g\in\Gamma}$ forms a bounded homogeneous orthonormal basis of normalizers for $P \subseteq N$. By Proposition 2.5.1, $N \cong P \rtimes_{\alpha,\omega'} \Gamma$ where ω' is the normalized 2-cocycle given by $\omega'(g,h) = v_g v_h v_{gh}^*$. Then $\nu(g,h) = \omega(g,h)\omega'(g,h)^*$ has $\sup_{g,h\in\Gamma} \|\nu(g,h) - I_P\| < 3(8 + 6\sqrt{2})\beta$ following the argument of (6.6). Thus, defining $\psi = -i \log \nu$, we obtain a bounded 2-cocycle in $Z_b^2(\Gamma, \mathcal{Z}(P)_{sa})$ by Lemma 2.5.2 (i) and the estimate $\|\psi\| \leq 2\sin^{-1}(3(8 + 6\sqrt{2})\beta/2)$ follows from the relation $|1 - e^{it}| = 2|\sin(t/2)|$. Note that $3(4 + 3\sqrt{2})\beta < 0.53$. For $0 \leq t \leq 0.53$, the convexity of $\sin^{-1}(t)$ yields $\sin^{-1}(t) \leq (\sin^{-1}(0.53)/0.53))t$, from which $\sin^{-1}(t) \leq 3t/(2\sqrt{2})$ follows by direct calculation. By hypothesis, $\psi = \partial\phi$ for some $\phi \in C_b^1(\Gamma, \mathcal{Z}(P)_{sa})$ with

$$\|\phi\| \leq K\|\psi\| \leq 2K \sin^{-1}(3(4 + 3\sqrt{2})\beta) \leq 9K(4 + 3\sqrt{2})\beta/\sqrt{2} = 9K(3 + 2\sqrt{2})\beta. \tag{6.12}$$

Lemma 2.5.2 (ii) gives $\nu(g,h) = e^{i\partial\phi(g,h)}$. We obtain the estimate

$$\|I_P - e^{i\phi(g)}\| \leq 9K(3 + 2\sqrt{2})\beta, \quad g \in \Gamma, \tag{6.13}$$

from $|1 - e^{it}| = 2|\sin(t/2)| \leq |t|$, so defining $v_g' = e^{i\phi(g)}v_g$, we have

$$\begin{aligned} \|v_g' - u_g\| &\leq \|v_g - u_g\| + \|e^{i\phi(g)} - I\| \\ &\leq (8 + 6\sqrt{2})\beta + 9K(3 + 2\sqrt{2})\beta, \quad g \in \Gamma. \end{aligned} \tag{6.14}$$

As in Proposition 2.5.1 (ii), the unitaries $(v_g')_{g\in\Gamma}$ also satisfy $v_g' x {v_g'}^* = \alpha_g(x)$ for $x \in P$ and since $v_g' v_h' {v_{gh}'}^* = \omega(g,h)$, it follows that N is isomorphic to $P \rtimes_{\alpha,\omega} \Gamma = M$. Further, the construction of Proposition 2.5.1 (i) gives an isomorphism $\theta : M = P \rtimes_{\alpha,\omega} \Gamma \to N$ with $\theta(x) = x$ for $x \in P$ and $\theta(u_g) = v_g'$ for $g \in \Gamma$.

Now M and N are both in standard position on $\mathcal{K}$ and $\beta < 1/47$ so Lemma 4.1.4 shows that ξ is also a tracial vector for N and N'. This allows us to define a unitary $W \in \mathcal{B}(\mathcal{K})$ by

$W(m\xi) = \theta(m)\xi$ for $m \in M$ and it is easy to check that $\theta(m) = WmW^*$ for $m \in M$. Note that

$$xWm\xi = x\theta(m)\xi = \theta(xm)\xi = Wxm\xi = Wx(m\xi), \quad x \in P,\ m \in M, \tag{6.15}$$

so that $W \in P'$. Similarly, as $\theta(M) = N \subseteq (J_M P J_M)'$,

$$\begin{aligned} J_M x J_M W m\xi &= J_M x J_M \theta(m)\xi = \theta(m) J_M x J_M \xi = \theta(m) x^* \xi \\ &= \theta(mx^*)\xi = Wmx^*\xi = W J_M x J_M m\xi, \quad x \in P,\ m \in M, \end{aligned} \tag{6.16}$$

so that $W \in (J_M P J_M)' = \langle M, e_P\rangle$. In this way $W \in P' \cap \langle M, e_P\rangle$.

For $g \in \Gamma$, write $e_g = u_g e_P u_g^* = v_g' e_P {v_g'}^*$. Since $P' \cap M \subseteq P$ (as α is centrally ergodic and properly outer), the projections e_g are central in $P' \cap \langle M, e_P\rangle$ by [35, Lemma 3.2] and $(P' \cap \langle M, e_P\rangle)e_g = \mathcal{Z}(P)e_g$ by [35, Lemma 3.4]. Therefore, we can write $W = \sum_{g\in\Gamma} w_g e_g$ for some unitaries $w_g \in \mathcal{Z}(P)$ and with strong*-convergence.

For each $h \in \Gamma$, we have $Wu_hW^* = \theta(u_h) = v_h'$ so that

$$\|W - u_h W u_h^*\| = \|W u_h W^* - u_h\| = \|u_h - v_h'\| \le (8 + 6\sqrt{2})\beta + 9K(3 + 2\sqrt{2})\beta. \tag{6.17}$$

Since

$$\begin{aligned} u_h W u_h^* &= u_h \left(\sum_{g\in\Gamma} w_g e_g\right) u_h^* = \sum_{g\in\Gamma} \alpha_h(w_g) u_h e_g u_h^* \\ &= \sum_{g\in\Gamma} \alpha_h(w_g) e_{hg} = \sum_{g\in\Gamma} \alpha_h(w_{h^{-1}g}) e_g, \end{aligned} \tag{6.18}$$

we can estimate

$$\begin{aligned} \sup_{g\in\Gamma} \|\alpha_h(w_{h^{-1}g}) - w_g\| &= \|\sum_{g\in\Gamma} (\alpha_h(w_{h^{-1}g}) - w_g) e_g\| \\ &= \|W - u_h W u_h^*\| \le (8 + 6\sqrt{2})\beta + 9K(3 + 2\sqrt{2})\beta, \quad h \in \Gamma. \end{aligned} \tag{6.19}$$

Taking $g = h$ above, we get

$$\sup_{h\in\Gamma} \|\alpha_h(w_{1_\Gamma}) - w_h\| \le (8 + 6\sqrt{2})\beta + 9K(3 + 2\sqrt{2})\beta. \tag{6.20}$$

For any $z \in \mathcal{Z}(P)$, we have $J_M z^* J_M e_P = z e_P$. Hence,

$$\begin{aligned} J_M w_{1_\Gamma} J_M e_g &= J_M w_{1_\Gamma} J_M u_g e_P u_g^* = u_g J_M w_{1_\Gamma}^* J_M e_P u_g^* \\ &= u_g w_{1_\Gamma} e_P u_g^* = \alpha_g(w_{1_\Gamma}) e_g, \quad g \in \Gamma. \end{aligned} \tag{6.21}$$

Define a unitary $W_1 = J_M w_{1_\Gamma} J_M \in J_M \mathcal{Z}(P) J_M \subseteq P' \cap \langle M, e_P\rangle$. By the calculation above

$$W_1 = \sum_{g\in\Gamma} J_M w_{1_\Gamma} J_M e_g = \sum_{g\in\Gamma} \alpha_g(w_{1_\Gamma}) e_g, \tag{6.22}$$

so that $\|W_1 - W\| \le (8 + 6\sqrt{2})\beta + 9K(3 + 2\sqrt{2}\beta)$ by (6.20). Since $W_1 \in J_M \mathcal{Z}(P) J_M \subseteq J_M P J_M \subseteq N'$, we have $\theta = \mathrm{Ad}(W_1^* W)$. By defining $U = W_1^* W$, we have $\theta = \mathrm{Ad}(U)$ and $\|U - I_{\mathcal{K}}\| = \|W_1 - W\| \le (8 + 6\sqrt{2})\beta + 9K(3 + 2\sqrt{2})\beta$. $\square$

The reduction procedure of Section 4 can now be used to prove Part (3) of Theorem B.

Theorem 6.3.2. *Let $\alpha : \Gamma \curvearrowright P$ be a trace preserving, centrally ergodic and properly outer action of a countable discrete group Γ on a finite amenable von Neumann algebra P with separable predual. Suppose that $H^2_b(\Gamma, \mathcal{Z}(P)_{sa}) = 0$ and let K be the constant (depending only on the action α) given by the open mapping theorem such that (2.29) holds. Let $M = P \rtimes_{\alpha,\omega} \Gamma$. Then, given a faithful unital normal representation $M \subseteq \mathcal{B}(\mathcal{H})$ and another von Neumann algebra $N \subseteq \mathcal{B}(\mathcal{H})$ with $d(M, N) < \gamma < 5.77 \times 10^{-16}$, there is a *-isomorphism $\theta : M \to N$ with*

$$\|\theta(x) - x\| < 902\gamma + (17 + 12\sqrt{2} + 105K) \times 50948 \times (301\gamma)^{1/2}, \quad x \in M,\ \|x\| \leq 1. \tag{6.23}$$

Proof. Take such a crossed product M. Suppose that $M \subseteq \mathcal{B}(\mathcal{H})$ is a faithful normal nondegenerate representation and suppose that $N \subseteq \mathcal{B}(\mathcal{H})$ is another von Neumann algebra acting nondegenerately on $\mathcal{H}$ with $d(M, N) < \gamma < 5.77 \times 10^{-16}$. By Theorem 3.1.1 (i), there is a unitary $u \in (M \cup N)''$ with $\|u - I_{\mathcal{H}}\| \leq 150\gamma$ and $uPu^* \subseteq N$. Set $N_1 = u^*Nu$ so that $P \subseteq M \cap N_1$. Then $d(M, N_1) \leq 301\gamma$. By Lemma 4.2.3, we can find a Hilbert space $\mathcal{K}$ and faithful normal *-representations $\pi : M \to \mathcal{B}(\mathcal{K})$ and $\rho : N_1 \to \mathcal{B}(\mathcal{K})$ so that:

(i) $\pi(M)$ and $\rho(N_1)$ are in standard position on $\mathcal{K}$ with common tracial vector ξ for $\pi(M)$, $\pi(M)'$, $\rho(N_1)$, and $\rho(N_1)'$;

(ii) $\pi(M) \subset_\beta \rho(N_1)$ and $\rho(N_1) \subset_\beta \pi(M)$ for some β satisfying
$$\beta < 50948 \times (301\gamma)^{1/2} < 1/47;$$

(iii) Whenever $x \in M$ and $y \in N_1$ have $\|x\|, \|y\| \leq 1$ and $\|x - y\| < \delta$ for some $\delta > 0$, it follows that $\|\pi(x) - \rho(y)\| \leq \delta + \beta$.

(iv) $\pi|_P = \rho|_P$;

(v) $\langle \pi(M), e_{\pi(P)} \rangle = \langle \rho(N_1), e_{\pi(P)} \rangle$, where these basic constructions are performed with respect to ξ.

Condition (v) ensures that, working with the modular conjugation operators induced by ξ, we have $J_{\pi(M)}\pi(P)J_{\pi(M)} = J_{\rho(N_1)}\pi(P)J_{\rho(N_1)} \subseteq \rho(N_1)'$. Then conditions (i), (ii) and (iv) allow us to apply Lemma 6.3.1 to $\pi(M)$ and $\rho(N_1)$ on $\mathcal{K}$. Consequently there is a unitary $U \in \pi(P)' \cap \langle \pi(M), e_{\pi(P)} \rangle$ such that $U\pi(M)U^* = \rho(N_1)$ and $\|U - I_{\mathcal{K}}\| \leq (8 + 6\sqrt{2})\beta + 9K(3 + 2\sqrt{2})\beta$. Define an isomorphism $\theta_1 : M \to N_1$ by $\rho^{-1} \circ \mathrm{Ad}(U) \circ \pi$. Given $x \in M$ with $\|x\| \leq 1$, fix $y \in N_1$ with $\|x - y\| \leq 301\gamma$. Then $\|\pi(x) - \rho(y)\| \leq 301\gamma + \beta$ so that

$$\begin{aligned}
\|\theta_1(x) - y\| &= \|U\pi(x)U^* - \rho(y)\| \leq 2\|U - I_{\mathcal{K}}\| + \|\pi(x) - \rho(y)\| \\
&\leq (16 + 12\sqrt{2} + (54 + 36\sqrt{2})K)\beta + 301\gamma + \beta \\
&< (17 + 12\sqrt{2} + 105K)\beta + 301\gamma. \qquad (6.24)
\end{aligned}$$

Finally, define an isomorphism $\theta : M \to N$ by $\theta = \mathrm{Ad}(u) \circ \theta_1$. This satisfies

$$\begin{aligned}
\|\theta(x) - x\| &\leq 2\|u - I_{\mathcal{H}}\| + \|\theta_1(x) - y\| + \|y - x\| \\
&< 902\gamma + (17 + 12\sqrt{2} + 105K)\beta \\
&< 902\gamma + (17 + 12\sqrt{2} + 105K) \times 50948 \times (301\gamma)^{1/2}, \quad x \in M,\ \|x\| \leq 1, \qquad (6.25)
\end{aligned}$$

where $y \in N_1$ with $\|y\| \leq 1$ is chosen such that $\|x - y\| \leq 301\gamma$. This completes the proof. □

The collection of groups all of whose actions satisfy the hypotheses of Theorem 6.3.2 contains $SL_n(\mathbb{Z})$ for $n \geq 3$ (see subsection 2.7). In particular, for $n \geq 3$, crossed products of the form $L^\infty(X, \mu) \rtimes_\alpha SL_n(\mathbb{Z})$ arising from free, ergodic, probability measure preserving actions have the property that sufficiently close algebras are isomorphic via an isomorphism close to the inclusion map. There is a family with cardinality that of the continuum of pairwise nonisomorphic factors arising in this fashion (see Remark 6.3.6 below).

Corollary 6.3.3. *For $n \geq 3$, let $\alpha : SL_n(\mathbb{Z}) \curvearrowright P$ be a centrally ergodic, properly outer trace preserving action on a finite amenable von Neumann algebra P with separable predual. For each $\varepsilon > 0$, there exists $\delta > 0$ with the following property: given a unital normal representation $\iota : P \rtimes_\alpha \Gamma \to \mathcal{B}(\mathcal{H})$ and a II_1 factor $N \subseteq \mathcal{B}(\mathcal{H})$ with $d(\iota(P \rtimes_\alpha \Gamma), N) < \delta$, there exists a surjective *-isomorphism $\theta : P \rtimes_\alpha \Gamma \to N$ with $\|\theta - \iota\| < \varepsilon$.*

We now turn to examples of nonamenable II_1 factors which satisfy the strongest form of the Kadison-Kastler conjecture and prove Theorem A. Such factors must inevitably have the similarity property [11]. Due to the presence of property T, we cannot construct crossed product factors $P \rtimes_\alpha SL_n(\mathbb{Z})$ for $n \geq 3$ with property Gamma (see Section 6.2 above), so we obtain the centralizing sequences which give the similarity property by tensoring these crossed product factors with the hyperfinite II_1 factor.

Theorem 6.3.4. *Let $\alpha : \Gamma \curvearrowright P_0$ be a trace preserving, centrally ergodic and properly outer action of a countable discrete group Γ on a finite amenable von Neumann algebra P_0 with separable predual and suppose that $H^2_b(\Gamma, \mathcal{Z}(P_0)_{sa}) = 0$. Let $M = (P_0 \rtimes_\alpha \Gamma) \overline{\otimes} R$, where R denotes the hyperfinite II_1 factor. Then M is strongly Kadison-Kastler stable.*

Precisely, let K be the constant (depending only on the action α) given by the open mapping theorem such that (2.29) holds for $\psi \in Z^2_b(\Gamma, \mathcal{Z}(P)_{sa})$. If $M \subseteq \mathcal{B}(\mathcal{H})$ is a unital normal representation of M and $N \subseteq \mathcal{B}(\mathcal{H})$ is another von Neumann algebra acting on $\mathcal{H}$ with $d(M, N) < \gamma$, where γ satisfies the inequality

$$(9356605 + 28918575K)\gamma < 2/5, \tag{6.26}$$

then there exists a unitary $U \in \mathcal{B}(\mathcal{H})$ with $UMU^* = N$ and

$$\|U - I_{\mathcal{H}}\| \leq (33080745 + 102242603K)\gamma. \tag{6.27}$$

Proof. Write $M_0 = P_0 \rtimes_\alpha \Gamma$ so that $M = M_0 \overline{\otimes} R$. Now suppose that M is represented as a unital von Neumann subalgebra of $\mathcal{B}(\mathcal{H})$ and N is another von Neumann algebra on $\mathcal{H}$ with $d(M, N) < \gamma$, where γ satisfies (6.26). For each 1-cochain $\phi : \Gamma \to \mathcal{Z}(P_0)_{sa}$, the inequality $\|\partial\phi\| \leq 3\|\phi\|$ implies that $K \geq 1/3$, so $\gamma < 1/(305 \times 903 \times 47)$ follows from (6.26) and the hypotheses of Lemma 5.2.4 are satisfied. Thus there exists a unitary $u \in (M \cup N)''$ with $\|u - I_{\mathcal{H}}\| \leq 150\gamma$ such that $uRu^* \subseteq N$, $P_0 \subseteq R' \cap u^*Nu$ and N is generated by the subfactors $(uRu^*)' \cap N$ and uRu^*. In particular $N \cong ((uRu^*)' \cap N) \overline{\otimes} uRu^*$ by [67] and so N is McDuff. Since M and N have property Gamma, Proposition 2.2.4 (ii) gives the near inclusions $M \subseteq_{cb,5\gamma} N$ and $N \subseteq_{cb,5\gamma} M$.

Write $N_1 = u^*Nu$ and $P = (P_0 \cup R)''$ so that $P \subseteq N_1 \cap M$ and N_1 is generated by the commuting algebras $R' \cap N_1$ and R on $\mathcal{H}$. Since $M \subseteq_{cb,305\gamma} N_1$ and $N_1 \subseteq_{cb,305\gamma} M$, Lemma 2.4.5 (ii) gives $M_0 = R' \cap M \subseteq_{cb,305\gamma} R' \cap N_1$ and $R' \cap N_1 \subseteq_{cb,305\gamma} M_0$. Proposition 2.2.3 induces the near inclusions $M' \subseteq_{cb,305\gamma} N_1'$ and $N_1' \subseteq_{cb,305\gamma} M'$. By construction $P_0 \subseteq M_0 \cap (R' \cap N_1)$.

By Theorem 4.3.3, applied with $\gamma_1 = 305\gamma$ replacing γ (valid as $305\gamma < 1/(903 \times 47)$), there exist representations $\pi : M \to \mathcal{B}(\mathcal{K})$ and $\rho : N_1 \to \mathcal{B}(\mathcal{K})$ which agree on P and such that:

(i) there is a tracial vector $\xi \in \mathcal{K}$ for $\pi(M)$, $\pi(M)'$, $\rho(N_1)$ and $\rho(N_1)'$;
(ii) $J_{\pi(M)}\pi(P)J_{\pi(M)} \subseteq \rho(N_1)'$;

(iii) $\pi(M) \subseteq_\beta \rho(N_1)$ and $\rho(N_1) \subseteq_\beta \pi(M)$ for $\beta = 903\gamma_1 = 275415\gamma$;
(iv) whenever $x \in M$ and $y \in N_1$ satisfy $\|x\|, \|y\| \leq 1$ and $\|x - y\| \leq \delta$, it follows that $\|\pi(x) - \rho(y)\| \leq \delta + 903\gamma_1$.

Since $\pi(M)$ is in standard position on $\mathcal{K}$ with tracial vector ξ, uniqueness up to spatial isomorphism allows us to assume that $\mathcal{K}$ factorizes as $\mathcal{K}_1 \otimes \mathcal{K}_2$, where $\mathcal{K}_1 = \overline{\pi(M_0 \otimes I_R)\xi}$ and $\mathcal{K}_2 = \overline{\pi(I_{M_0} \otimes R)\xi}$ and with the following additional properties. The vector ξ factorizes as $\xi_1 \otimes \xi_2$, $\pi(M_0)$ acts in standard position on $\mathcal{K}_1$ with respect to ξ_1 and $\pi(R)$ acts in standard position on $\mathcal{K}_2$ with respect to ξ_2. Consequently, with respect to this factorization, $\pi(R)' \cap (J_{\pi(M)}\pi(R)J_{\pi(M)})' = \mathcal{B}(\mathcal{K}_1) \otimes \mathbb{C}I_{\mathcal{K}_2}$. Since $J_{\pi(M)}\pi(R)J_{\pi(M)} \subseteq \rho(N_1)'$, we have $\pi(R)' \cap \rho(N_1) \subseteq \pi(R)' \cap (J_{\pi(M)}RJ_{\pi(M)})' = \mathcal{B}(\mathcal{K}_1) \otimes \mathbb{C}I_{\mathcal{K}_2}$ and so the factorization of $N_1 = N_0 \overline{\otimes} R$ respects the decomposition of $\mathcal{K} = \mathcal{K}_1 \otimes \mathcal{K}_2$.

It follows that $\pi(M_0)$ and $\rho(N_0)$ can be regarded as represented on $\mathcal{K}_1$ where ξ_1 is a tracial vector for $\pi(M_0)$, $\pi(M_0)'$, $\rho(N_0)$, and $\rho(N_0)'$. Further, $\pi(P_0) \subseteq \pi(M_0) \cap \rho(N_0)$ and $J_{\pi(M_0)}\pi(P_0)J_{\pi(M_0)} \subseteq \rho(N_0)'$, where $J_{\pi(M_0)}$ is the modular conjugation operator on $\mathcal{K}_1$ defined with respect to ξ_1. Thus Lemma 6.3.1 gives a unitary $u_0 \in \mathcal{B}(\mathcal{K}_1)$ with $u_0\pi(M_0)u_0^* = \rho(N_0)$ and $\|u_0 - I_{\mathcal{K}_1}\| \leq (8 + 6\sqrt{2} + 9K(3 + 2\sqrt{2}))\beta$. Define $u_1 = u_0 \otimes I_{\mathcal{K}_2}$ so that u_1 is a unitary on $\mathcal{K}$ with $u_1\pi(M)u_1^* = \rho(N_1)$ and $\|u_1 - I_{\mathcal{K}}\| \leq (8 + 6\sqrt{2} + 9K(3 + 2\sqrt{2}))\beta$.

Define $\theta = \rho^{-1} \circ \mathrm{Ad}(u_1) \circ \pi : M \to N_1$. For a contraction $x \in M$, choose a contraction $y \in N_1$ with $\|x - y\| \leq 301\gamma$ (possible as $d(M, N) < \gamma$ and $\|u - I_{\mathcal{H}}\| \leq 150\gamma$). Estimating in a very similar fashion to the end of the proof of Theorem 6.3.2 shows that

$$\begin{aligned} \|\theta(x) - y\| &= \|u_1\pi(x)u_1^* - \rho(y)\| \leq 2\|u_1 - I_{\mathcal{H}}\| + \|\pi(x) - \rho(y)\| \\ &< (16 + 12\sqrt{2} + 105K)\beta + 301\gamma + \beta < (9356304 + 28918575K)\gamma. \end{aligned} \tag{6.28}$$

Thus

$$\|\theta(x) - x\| \leq \|\theta(x) - y\| + \|y - x\| \leq (9356605 + 28918575K)\gamma. \tag{6.29}$$

By hypothesis this last quantity is less than 2/5, so Lemma 2.2.6 (ii) applies. Therefore, there exists a unitary u_2 on $\mathcal{H}$ with $\theta = \mathrm{Ad}(u_2)$ and

$$\|u_2 - I_{\mathcal{H}}\| \leq 2^{-1/2} \times 5 \times (9356605 + 28918575K)\gamma \leq (33080595 + 102242603K)\gamma. \tag{6.30}$$

Write $U = uu_2$ so that $UMU^* = N$ and

$$\|U - I_{\mathcal{H}}\| \leq \|u_2 - I_{\mathcal{H}}\| + \|u - I_{\mathcal{H}}\| \leq (33080745 + 102242603K)\gamma, \tag{6.31}$$

establishing (6.27) as required. □

Theorem A now follows immediately from Theorem 6.3.4 and Theorem 2.7.1.

Corollary 6.3.5. *Let $n \geq 3$ and $\Gamma = SL_n(\mathbb{Z})$. Given any free, ergodic, measure preserving action $\alpha : \Gamma \curvearrowright (X, \mu)$ on a standard probability space, the II_1 factor $(L^\infty(X, \mu) \rtimes_\alpha \Gamma) \overline{\otimes} R$ is strongly Kadison-Kastler stable.*

Remark 6.3.6. There are a continuum of pairwise nonisomorphic II_1 factors to which the previous corollary applies. Let us specialize to the case when $P_0 = L^\infty(X, \mu)$ and the action α is a Bernoulli shift (meaning that (X, μ) arises from an infinite product $(X_0, \mu_0)^\Gamma = \{f : \Gamma \to X_0\}$ indexed by the group Γ, and Γ acts on X by translation: $(g \cdot f)(h) = f(g^{-1}h)$). In [83, 84] Popa established a breakthrough superrigidity result for a class of actions including Bernoulli actions $\Gamma \curvearrowright (X, \mu)$ of ICC groups with property T ([84, Theorem 0.1]): given two such actions $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \nu)$, if the crossed product factor $L^\infty(X, \mu) \rtimes \Gamma$ is isomorphic to an amplification $(L^\infty(Y,\nu)\rtimes\Lambda)_t$ for some $t>0$ (the amplification M^t of a II_1 factor is a factor $p(M\otimes\mathbb{M}_n)p$, where $n\in\mathbb{N}$ has $n>t$ and the normalized trace of the projection p in $M\otimes\mathbb{M}_n$ is t/n), then $t=1$, $\Gamma\cong\Lambda$ and the actions $\Gamma\curvearrowright(X,\mu)$ and $\Lambda\curvearrowright(Y,\nu)$ are conjugate. This result has recently been extended by Ioana [42] to allow $\Lambda\curvearrowright(Y,\nu)$ to be any free ergodic probability measure preserving action of a countable discrete group. At the level of actions, Bowen introduced in [4] an entropy invariant for measure preserving actions of sofic groups (a class of groups including all residually finite groups, and hence $SL_n(\mathbb{Z})$ for $n\geq 3$). One of many consequences of this work is that conjugacy of two Bernoulli shifts $\Gamma\curvearrowright(X_0,\mu_0)^\Gamma$ and $\Gamma\curvearrowright(Y_0,\nu_0)^\Gamma$ of a sofic group over base spaces with finite entropy implies that the entropies $H(X_0,\mu_0)$ and $H(Y_0,\nu_0)$ are equal ([4, Theorem 1.1]).

By considering a family of base spaces on which the entropy takes all values in $[0,\infty)$, the results discussed above show that there is a family $\mathcal{F}$ with cardinality equal to that of the continuum such that $\mathcal{F}$ consists of II_1 factors of the form $L^\infty(X,\mu)\rtimes_\alpha SL_3(\mathbb{Z})$ arising from Bernoulli shifts such that no two are isomorphic even after amplification. Since $SL_3(\mathbb{Z})$ has property T, these factors cannot have property Gamma. Given II_1 factors M_1 and M_2 without property Gamma, Theorem 5.1 of [85] shows that if $M_1\,\overline{\otimes}\,R\cong M_2\,\overline{\otimes}\,R$, then M_1 is isomorphic to an amplification of M_2. Consequently the elements of the family $\mathcal{F}^R=\{M\,\overline{\otimes}\,R : M\in\mathcal{F}\}$ are pairwise nonisomorphic.

The following result is obtained in exactly the same way as the deduction of Theorem 6.3.4 from Lemma 6.3.1. We omit the details.

Proposition 6.3.7. *Suppose that $M\subseteq\mathcal{B}(L^2(M))$ is a II_1 factor with the property that for $\varepsilon>0$, there exists $\delta>0$ such that given a II_1 factor $N\subseteq\mathcal{B}(L^2(M))$ with $d(M,N)<\delta$, then there is a unitary $u\in\mathcal{B}(L^2(M))$ with $uMu^*=N$ and $\|u-I_{L^2(M)}\|\leq\varepsilon$. Then $M\,\overline{\otimes}\,R$ is strongly Kadison Kastler stable.*

6.4. **Type II_1 Measured equivalence relations.** In their seminal papers [37, 38] Feldman and Moore assign cohomology groups to measurable equivalence relations and generalize the twisted crossed product construction to produce an inclusion of a Cartan masa inside a von Neumann algebra from a countable measured equivalence relation and (the orbit of) a certain element of 2-cohomology. Conversely, they show that an inclusion $A\subseteq M$ of a Cartan masa inside a von Neumann algebra is determined by this data. In this subsection, we work in this more general context. We show that close inclusions of Cartan masas in II_1 factors give rise to isomorphic relations and can be represented by uniformly close cocycles.

We first recall from [38] how to associate a measurable equivalence relation to an inclusion $A\subseteq M$ of a Cartan masa in a II_1 factor ([38] works in the context of arbitrary von Neumann algebras, but we restrict to the factor case for simplicity and the finite case so we can apply the results of Section 4). First we place M in standard position on $\mathcal{H}$ and let J_M be the modular conjugation operator. Then identify A with $L^\infty(X,\mu)$, where the measure μ arises from the trace on M. The algebra $B=J_MAJ_M$ commutes with A and so A and B together generate an abelian von Neumann algebra $C=(A\cup B)''$ on $\mathcal{H}$. We may write $C=L^\infty(R,\nu)$ for some measure space (R,ν). The inclusions $A\hookrightarrow C$ and $B\hookrightarrow C$ induce two Borel maps $\pi_l,\pi_r:R\to X$ and these can be used to view R as a Borel subset of $X\times X$, using π_l and π_r as the projections. Propositions 3.2-3.9 of [38] then show that this procedure does produce a countable standard relation R which we call the equivalence relation of the inclusion $A\subseteq M$ (and the counting measure induced on this relation is in the same measure class as the initially chosen ν).

Proposition 6.4.1. *Let M and N be* II_1 *factors acting nondegenerately on* $\mathcal{H}$ *with* $M \subset_\gamma N \subset_\gamma M$ *for some* $\gamma < 5.7 \times 10^{-16}$. *Suppose that* $A \subseteq M$ *is a Cartan masa in* M. *Then any von Neumann subalgebra* $B \subseteq N$ *with* $A \subset_\delta B \subset_\delta A$ *for* $\delta < 1/101$ *is a Cartan masa and the* II_1 *measurable equivalence relations determined by the inclusions* $A \subseteq M$ *and* $B \subset N$ *are equivalent.*

Proof. By [21, Corollary 4.4], there is a unitary $u \in (A \cup B)''$ with $uAu^* \subseteq B$, $\|u - I_{\mathcal{H}}\| \leq 150\gamma$ and $\|uxu^* - x\| \leq 100\gamma\|x\|$ for $x \in A$. Set $N_1 = u^*Nu$ so that $A \subset M \cap N_1$ and $M \subset_{\gamma_1} N_1 \subset_{\gamma_1} M$, where $\gamma_1 = 301\gamma$. Lemma 4.2.3 applies to M and N_1 with $P = A$, so that M and N_1 can be simultaneously represented in standard position on the same Hilbert space $\mathcal{K}$ so that $J_M A J_M = J_{N_1} A J_{N_1}$. Now when we construct the equivalence relations from $(A \cup J_M A J_M)''$ and $(A \cup J_N A J_N)''$ as described immediately before the lemma, we get precisely the same orbit equivalence relation for the inclusions $A \subseteq M$ and $A \subseteq N_1$. Since the inclusion $A \subseteq N_1$ is unitarily equivalent to $B \subseteq N$, it follows that $B \subseteq N$ and $A \subseteq M$ induce isomorphic equivalence relations. □

We now turn to the 2-cohomology class induced by the inclusion $A \subseteq M$ of a Cartan masa in a II_1 factor. Let R be a countable measured equivalence relation on (X, μ) equipped with the measure class of the counting measure (see [37, Theorem 2]). The cohomology group we need is $H^2(R, \mathbb{T})$. These groups were introduced in [37, Section 6] and we briefly review how they are defined. For $n \geq 1$, let $R^n = \{(x_0, \dots, x_n) \in X^{n+1} : (x_{i-1}, x_i) \in R, \quad i = 1, \dots, n\}$. This is a Borel subset of X^{n+1} and is equipped with a measure class generalizing the counting measure class on R to R^n (see [37, Proposition 2.3]). We then define the abelian group of n-cochains, $C^n(R, \mathbb{T})$ to be equivalence classes of Borel functions $R^n \to \mathbb{T}$ with pointwise multiplication. The coboundary map $\partial_n : C^n \to C^{n+1}$ is given by

$$(\partial f)(x_0, \dots, x_{n+1}) = \prod_{i=0}^{n+1} f(x_0, \dots, x_{i-1}, \widehat{x_i}, x_{i+1}, \dots, x_{n+1})^{(-1)^i}, \tag{6.32}$$

where $(x_0, \dots, x_{i-1}, \widehat{x_i}, x_{i+1}, \dots, x_{n+1})$ denotes the $n+1$-tuple in R^n obtained by omitting x_i. Since $\partial_n \partial_{n-1} = 0$, it makes sense to define the cocycles by $Z^n(R, \mathbb{T}) = \ker \partial_n$, the coboundaries by $B^n(R, \mathbb{T}) = \mathrm{Im}\ \partial_{n-1}$ and the n-th cohomology group by $H^n(R, \mathbb{T}) = Z^n(R, \mathbb{T})/B^n(R, \mathbb{T})$.

The automorphism group $\mathrm{Aut}(R)$ of the relation R consists of the (equivalence classes of) Borel isomorphisms $\theta : X \to X$ for which $(\theta \times \theta)(R) = R$. We have an action of $\mathrm{Aut}(R)$ on the cochains $C^n(R, \mathbb{T})$ given by

$$(\theta f)(x_0, \dots, x_n) = f(\theta^{-1}(x_0), \dots, \theta^{-1}(x_n)), \quad (x_0, \dots, x_n) \in R^n, \tag{6.33}$$

which induces an action on the cohomology groups $H^n(R, \mathbb{T})$. The main result of [38] is that an inclusion of a Cartan masa in a von Neumann algebra is completely classified by the induced equivalence relation R and the orbit under $\mathrm{Aut}(R)$ in $H^2(R, \mathbb{T})$ of a cohomology class constructed from the original inclusion. To give a version of Lemma 6.1.1 in this setting we need to review how this construction works in detail.

Feldman and Moore observe that when A is Cartan, $L^\infty(R, \nu) = (A \cup J_M A J_M)''$ is maximal abelian in $\mathcal{B}(\mathcal{H})$ (see [38, Proposition 3.11] and also [81]). This enables them to view $\mathcal{H}$ as $L^2(R, \nu)$ and gives rise to a decomposition of normalizers in [38, Proposition 3.10] as follows. Given any normalizer $u \in \mathcal{N}(A \subseteq M)$ which induces the corresponding action θ_u on X (meaning that $(ufu)^*(x) = f(\theta_u^{-1}(x))$ for $f \in L^\infty(X)$ and $x \in X$), there is a measurable map $a_u : R \to \mathbb{T}$ such that, viewing u as an element of $\mathcal{B}(L^2(R,\nu))$, we have

$$(u\psi)(x,y) = a_u(x,y)\psi(\theta_u^{-1}(x),y), \quad \psi \in L^2(R,\nu),\ (x,y) \in R. \tag{6.34}$$

This is proved by defining another unitary u' on $L^2(R,\nu)$ by $(u'\psi)(x,y) = \psi(\theta_u^{-1}(x),y)$ and noting that both u and u' normalize $C = (A \cup J_M A J_M)''$ and $ucu^* = u'cu'^*$ for all $c \in C$. Since $C = (A \cup J_M A J_M)'' = L^\infty(R,\nu)$ is maximal abelian in $\mathcal{B}(L^2(R,\nu))$ it follows that uu'^* is a unitary, say a_u, in $L^\infty(R,\nu)$ and the claim follows.

Let G be a dense subgroup of $\mathcal{N}(A \subseteq M)$ which contains $\mathcal{U}(A)$ and for which the quotient group $G_0 = G/\mathcal{U}(A)$ is countable. Two normalizers u_1, u_2 in the same $\mathcal{U}(A)$-coset induce the same action so that $\theta_{u_1} = \theta_{u_2}$, and consequently G_0 acts on X, inducing an orbit equivalence relation R_{G_0}. In [38, Proposition 3.9] it is shown that the equivalence class of this relation does not depend on the choice of G and that it is equivalent to the original orbit equivalence relation R.

Fix such a group G and let D consist of one representative in G of each coset of $G_0 = G/\mathcal{U}(A)$. We may assume that $1 \in D$ and if $u \in D$, then $u^* \in D$. If $(x,y) \in R_{G_0}$, then we can find at least one element u of D with $\theta_u(x) = y$; since the action is not necessarily free this u need not be unique. To resolve this, Feldman and Moore partition R into a disjoint union of Borel sets $(R(u))_{u \in D}$ so that $R(1)$ is the diagonal, and $R(u^*)$ is obtained by flipping $R(u)$. They then define a Borel function $c : R^2 \to \mathbb{T}$ by

$$c(x,y,z) = a_u(x,y), \quad \text{where } (x,z) \in R(u), \tag{6.35}$$

([38, Proposition 3.13] shows that this is well defined). Implicit here is the fact that while R and R_{G_0} are equivalent, they are not equal: thus one deletes a null set where no such u can be found. Finally, they obtain a cocycle t on R^2 by showing that the quantity

$$t(x,y,z) = c(x,y,r)c(y,z,r)c(x,z,r)^{-1}, \quad (x,y,z) \in R^2 \tag{6.36}$$

is independent of the variable r and defines a 2-cocycle. Further, while t itself depends on the choice of D above, its class in $H^2(R,\mathbb{T})$ does not. This class only depends on $A \subseteq M$ and the identification of A with $L^\infty(X)$ (this last point explains why Feldman and Moore's invariant is the orbit of the class of t under $\mathrm{Aut}(R)$). We are now in a position to establish the II_1 measured equivalence relation version of Lemma 6.1.1. We state the result in the context of a common Cartan masa $A \subseteq M$ and $A \subseteq N$; the standard reduction procedure of first twisting N so that it also contains A can be used in the general case.

Proposition 6.4.2. *Let M and N be II_1 factors acting nondegenerately on $\mathcal{H}$ with $d(M,N) < \gamma < 1.74 \times 10^{-13}$. Suppose that $A \subseteq M \cap N$ is a Cartan masa in both M and N. Then the inclusions $A \subseteq M$ and $A \subseteq N$ determine isomorphic measurable equivalence relations. Denote the class of these relations as R. Then we can choose representatives ω_M and ω_N in $Z^2(R,\mathbb{T})$ of the (orbits of the) Feldman-Moore cohomology classes corresponding to $A \subseteq M$ and $A \subseteq N$ such that*

$$\|\omega_M - \omega_N\|_{L^\infty(R^2)} \leq 3242311\gamma^{1/2}. \tag{6.37}$$

Proof. Write $A = L^\infty(X,\mu)$, where μ is the measure on X obtained from restricting τ_M to A. The bound on γ is chosen so that Lemma 4.2.3 applies. Thus we can replace $\mathcal{H}$ by the $\mathcal{K}$ of that lemma and assume that M and N are simultaneously in standard position with the same fixed tracial vector on $\mathcal{K}$ so that $J_M A J_M = J_N A J_N$, at the expense of changing the distance estimate to $M \subset_\beta N \subset_\beta M$, where $\beta = 50948\gamma^{1/2}$. In this way we obtain a common algebra $(A \cup J_M A J_M)'' = (A \cup J_N A J_N)''$ which we identify as $L^\infty(R,\nu)$, for some measurable II_1 equivalence relation R on X. Further, we identify $\mathcal{K}$ with $L^2(R,\nu)$.

Theorem 3.2.3 gives us a well defined group isomorphism $\Theta : \mathcal{N}(A \subseteq M)/\mathcal{U}(A) \to \mathcal{N}(A \subseteq N)/\mathcal{U}(A)$ obtained by $\Theta(u\mathcal{U}(A)) = v\mathcal{U}(A)$, where $v \in \mathcal{N}(A \subseteq N)$ satisfies $\|v-u\| < 15\sqrt{2}\beta$. Choose a dense subgroup $\tilde{G}$ of $\mathcal{N}(A \subset M)$ which contains $\mathcal{U}(A)$ so that $\tilde{G}_0 = \tilde{G}/\mathcal{U}(A)$ is countable. Define $\tilde{H} = \Theta(\tilde{G}_0)$ and let H be any dense subgroup of $\mathcal{N}(A \subseteq N)$ with $\mathcal{U}(A) \subseteq H$ so that $H_0 = H/\mathcal{U}(A)$ is countable and contains $\tilde{H}$. Let D be a set consisting of one representative in H of each coset in H_0 so that $1 \in D$ and D is closed under the adjoint operation. For each $v \in D$, choose a unitary normalizer $u_v \in \mathcal{N}(A \subseteq M)$ with $\|u_v - v\| < 15\sqrt{2}\beta$ by Lemma 3.2.1 (ii), the bound on γ being small enough to ensure that $2\sqrt{2}\beta < 1/8$ so that this lemma applies. We insist that these choices are made such that $u_1 = 1$ and $(u_v)^* = u_{v^*}$ for each $v \in D$. Now let G be the subgroup of $\mathcal{N}(A \subseteq M)$ generated by $\mathcal{U}(A)$ and $\{u_v : v \in D\}$. Note that $\Theta^{-1}(H_0) \supseteq \Theta^{-1}(\tilde{H}) = \tilde{G}_0$ so that G contains $\tilde{G}$ and hence is dense in $\mathcal{N}(A \subseteq M)$. Since D is countable, $G_0 = G/\mathcal{U}(A)$ is countable. By construction, the set $E = \{u_v : v \in D\}$ consists of exactly one representative in G of each coset of G_0.

Each $u_v \in E$, arises from a measurable function $a_{u_v} : R \to \mathbb{T}$ from (6.34), so that

$$(u_v\psi)(x,y) = a_{u_v}(x,y)\psi(\theta_{u_v}^{-1}(x),y), \quad \psi \in L^2(R,\nu), \quad (x,y) \in R, \tag{6.38}$$

where θ_{u_v} is the action induced on X via the normalizer u_v of $L^\infty(X) = A$. Similarly for $v \in D$, we obtain a measurable function $b_v : R \to \mathbb{T}$ such that

$$(v\psi)(x,y) = b_v(x,y)\psi(\theta_v^{-1}(x),y), \quad \psi \in L^2(R,\nu), \quad (x,y) \in R. \tag{6.39}$$

As u_v and v induce the same action on A, it follows that $\theta_{u_v} = \theta_v$ and hence

$$\|a_{u_v} - b_u\|_{L^\infty(R)} = \|u_v - v\|_{\mathcal{B}(L^2(R,\nu))} \leq 15\sqrt{2}\beta. \tag{6.40}$$

Now choose a Borel partition $(R(v))_{v\in D}$ of R so that $R(1)$ is the diagonal and $R(v^*)$ is the flip of $R(v)$ for all $v \in D$. We can view this as a partition indexed by E as well. Thus when we define $c : R^2 \to \mathbb{T}$ for $A \subseteq M$ by $c(x,y,z) = a_{u_v}(x,y)$ when $(x,z) \in R(u_v)$ and $d : R^2 \to \mathbb{T}$ for $A \subseteq N$ by $d(x,y,z) = b_v(x,y)$ when $(x,z) \in R(v)$, it follows that $\|c-d\|_{L^\infty(R^2)} \leq 15\sqrt{2}\beta$. Now define a representative ω_M of the Feldman-Moore cococycle by (6.36) for $A \subseteq M$ and a representative ω_N for the cocycle of $A \subseteq N$ by the analogous formula using d in place of c. It follows that

$$\|\omega_M - \omega_N\|_{L^\infty(R^2)} \leq 3\|c-d\|_{L^\infty(R^2)} \leq 45\sqrt{2}\beta < 3242311\gamma^{1/2}, \tag{6.41}$$

as claimed. □

In just the same way as Corollary 6.1.2 is deduced from Theorem 6.1.1, it follows that a II_1 factor M arising from the Feldman-Moore construction from the equivalence relation R is weakly Kadison-Kastler stable when the natural comparison map

$$H_b^2(R,\mathbb{R}) \to H^2(R,\mathbb{R})$$

vanishes.

6.5. **Concluding remarks.** In general terms, we have fixed a von Neumann algebra M in a certain class and asked whether it is isomorphic to all of its close neighbors N. We end by setting out what our methods give when restrictions are imposed on both M and N. In the next result, we use the fact that N is assumed to already be presented as a genuine crossed product to avoid using cohomological hypotheses to untwist the cocycle in Theorem 6.1.1.

Corollary 6.5.1. *Let $\alpha : \Gamma \curvearrowright (X,\mu)$ and $\beta : \Lambda \curvearrowright (Y,\nu)$ be free, ergodic, probability measure preserving actions of countable discrete groups Γ and Λ. Let $M = L^\infty(X,\mu) \rtimes_\alpha \Gamma$, $N = L^\infty(Y,\nu) \rtimes_\beta \Lambda$ and suppose that M contains a unique Cartan masa up to unitary conjugacy. Suppose that M and N are represented as von Neumann algebras on $\mathcal{H}$ with $d(M,N) < 5.8 \times 10^{-16}$. Then M is isomorphic to N.*

Proof. Theorem 6.1.1 shows that we can identify N as a twisted crossed product $L^\infty(X,\mu) \rtimes_{\alpha,\omega} \Gamma$ for some 2-cocycle $\omega \in Z^2(\Gamma, \mathcal{U}(L^\infty(X,\mu)))$. Since M contains a unique Cartan masa, so too does N by Theorem 4.2.5 (iv). By the uniqueness of the Cartan masa in N, the orbit equivalence relations induced by α and β must be isomorphic. Since the isomorphism class of a crossed product depends only on the orbit equivalence relation of the underlying action, it follows that $N \cong M$. □

By [88, Theorem 1.1], the factor M is guaranteed to have a unique Cartan masa in the following circumstances: Γ is a nonelementary hyperbolic group; a lattice in a connected noncompact rank one simple Lie group with finite center; a limit group or a finite product of these groups. We do not know how to extend the class of weakly Kadison-Kastler factors to reach all crossed products containing a unique Cartan masa. In particular our strategy of relying on the vanishing of the comparison map

$$H^2_b(\Gamma, L^2(\mathcal{Z}(P)_{sa})) \to H^2(\Gamma, L^2(\mathcal{Z}(P)_{sa}))$$

cannot work in this level of generality: when Γ is a hyperbolic group the comparison map $H^2_b(\Gamma,\mathbb{R}) \to H^2(\Gamma,\mathbb{R})$ is always surjective [59].

References

[1] Charles A. Akemann and Martin E. Walter, *Unbounded negative definite functions*, Canad. J. Math. **33** (1981), no. 4, 862–871. 45
[2] Huzihiro Araki and Marie Choda, *Property T and actions on the hyperfinite* II_1*-factor*, Math. Japon. **28** (1983), no. 2, 205–209. 46
[3] William B. Arveson, *Interpolation problems in nest algebras*, J. Funct. Anal. **20** (1975), 208–233. 9, 11
[4] Lewis Bowen, *Measure conjugacy invariants for actions of countable sofic groups*, J. Amer. Math. Soc. **23** (2010), no. 1, 217–245. 2, 52
[5] ———, *Orbit equivalence, coinduced actions and free products*, Groups Geom. Dyn. **5** (2011), no. 1, 1–15. 45
[6] Kenneth S. Brown, *Cohomology of groups*, Graduate Texts in Mathematics, vol. 87, Springer-Verlag, New York, 1994, Corrected reprint of the 1982 original. 17
[7] John W. Bunce, *The similarity problem for representations of* C^**-algebras*, Proc. Amer. Math. Soc. **81** (1981), no. 3, 409–414. 5
[8] Marc Burger and Nicolas Monod, *Bounded cohomology of lattices in higher rank Lie groups*, J. Eur. Math. Soc. (JEMS) **1** (1999), no. 2, 199–235. 20
[9] ———, *Continuous bounded cohomology and applications to rigidity theory*, Geom. Funct. Anal. **12** (2002), no. 2, 219–280. 2, 20, 21
[10] Marc Burger, Narutaka Ozawa, and Andreas Thom, *On Ulam stability*, Israel J. Math., to appear. arXiv:1010.0565, 2010. 3

[11] Jan Cameron, Erik Christensen, Allan M. Sinclair, Roger R. Smith, Stuart White, and Alan D. Wiggins, *A remark on the similarity and perturbation problems*, C. R. Acad. Sci. Canada, in press. arXiv:1206.5405, 2012. 4, 5, 8, 37, 50

[12] Jan Cameron, Erik Christensen, Allan M. Sinclair, Roger R. Smith, Stuart White, and Alan D. Wiggins, *Type II_1 factors satisfying the spatial isomorphism conjecture*, Proc. Natl. Acad. Sci. USA., to appear, arXiv:1211.6963, 2012. 1, 2

[13] Pierre-Alain Cherix, Michael Cowling, Paul Jolissaint, Pierre Julg, and Alain Valette, *Groups with the Haagerup property*, Progress in Mathematics, vol. 197, Birkhäuser Verlag, Basel, 2001, Gromov's a-T-menability. 46

[14] Ionut Chifan and Thomas Sinclair, *On the structure theory of II_1 factors of negatively curved groups*, Ann. Sci. Éc. Norm. Supér, in press. arXiv:1103.4299, 2011. 6, 35, 38

[15] Marie Choda, *Some relations of* II_1*-factors on free groups*, Math. Japon. **22** (1977), no. 3, 383–394. 17

[16] Man Duen Choi and Erik Christensen, *Completely order isomorphic and close C^*-algebras need not be *-isomorphic*, Bull. London Math. Soc. **15** (1983), no. 6, 604–610. 2

[17] Erik Christensen, *Perturbations of type I von Neumann algebras*, J. London Math. Soc. (2) **9** (1974/75), 395–405. 2

[18] ______, *Perturbations of operator algebras*, Invent. Math. **43** (1977), no. 1, 1–13. 2, 3, 21, 23

[19] ______, *Perturbations of operator algebras II*, Indiana Univ. Math. J. **26** (1977), 891–904. 9, 10, 12, 15, 21, 22

[20] ______, *Subalgebras of a finite algebra*, Math. Ann. **243** (1979), no. 1, 17–29. 13

[21] ______, *Near inclusions of C^*-algebras*, Acta Math. **144** (1980), no. 3-4, 249–265. 2, 3, 4, 8, 11, 12, 21, 22, 53

[22] ______, *Extensions of derivations. II*, Math. Scand. **50** (1982), no. 1, 111–122. 4, 8, 9

[23] ______, *Similarities of* II_1 *factors with property* Γ, J. Operator Theory **15** (1986), no. 2, 281–288. 5

[24] ______, *Finite von Neumann algebra factors with property* Γ, J. Funct. Anal **186** (2001), no. 2, 366–380. 12

[25] Erik Christensen, Florin Pop, Allan M. Sinclair, and Roger R. Smith, *On the cohomology groups of certain finite von Neumann algebras*, Math. Ann. **307** (1997), no. 1, 71–92. 9

[26] Erik Christensen, Allan M. Sinclair, Roger R. Smith, and Stuart White, *Perturbations of C^*-algebraic invariants*, Geom. Funct. Anal. **20** (2010), no. 2, 368–397. 4, 5, 7, 10, 11, 12, 14, 25, 37

[27] Erik Christensen, Allan M. Sinclair, Roger R. Smith, Stuart White, and Wilhelm Winter, *The spatial isomorphism problem for close separable nuclear C^*-algebras*, Proc. Natl. Acad. Sci. USA **107** (2010), no. 2, 587–591. 2

[28] ______, *Perturbations of nuclear C^*-algebras*, Acta. Math. **208** (2012), no. 1, 93–150. 2, 3, 38

[29] Alain Connes and Vaughan F. R. Jones, *A* II_1 *factor with two nonconjugate Cartan subalgebras*, Bull. Amer. Math. Soc. (N.S.) **6** (1982), no. 2, 211–212. 35

[30] Alain Connes and Benjamin Weiss, *Property T and asymptotically invariant sequences*, Israel J. Math. **37** (1980), no. 3, 209–210. 46

[31] Kenneth R. Davidson, *The distance between unitary orbits of normal operators*, Acta Sci. Math. (Szeged) **50** (1986), no. 1-2, 213–223. 32

[32] Jacques Dixmier, *Sous-anneaux abéliens maximaux dans les facteurs de type fini*, Ann. of Math. (2) **59** (1954), 279–286. 19

[33] ______, *Les algèbres d'opérateurs dans l'espace hilbertien (Algèbres de von Neumann)*, Cahiers scientifiques, Fascicule XXV, Gauthier-Villars, Paris, 1957. 14, 36, 40

[34] Edward G. Effros, *Property* Γ *and inner amenability*, Proc. Amer. Math. Soc. **47** (1975), 483–486. 45

[35] Junsheng Fang, Roger R. Smith, Stuart White, and Alan D. Wiggins, *Groupoid normalizers of tensor products*, J. Funct. Anal. **258** (2010), no. 1, 20–49. 13, 48

[36] Ilijas Farah, Bradd Hart, and David Sherman, *Model theory of operator algebras III: Elementary equivalence and II_1 factors*, preprint, arXiv:1111.0998v1., 2011. 39

[37] Jacob Feldman and Calvin C. Moore, *Ergodic equivalence relations, cohomology, and von Neumann algebras. I*, Trans. Amer. Math. Soc. **234** (1977), no. 2, 289–324. 52, 53

[38] ______, *Ergodic equivalence relations, cohomology, and von Neumann algebras. II*, Trans. Amer. Math. Soc. **234** (1977), no. 2, 325–359. 6, 19, 52, 53, 54

[39] Uffe Haagerup, *Solution of the similarity problem for cyclic representations of C^*-algebras*, Ann. of Math. (2) **118** (1983), no. 2, 215–240. 5

[40] Richard H. Herman and Vaughan F. R. Jones, *Central sequences in crossed products*, Operator algebras and mathematical physics (Iowa City, Iowa, 1985), Contemp. Math., vol. 62, Amer. Math. Soc., Providence, RI, 1987, pp. 539–544. 45

[41] Cyril Houdayer, *Strongly solid group factors which are not interpolated free group factors*, Math. Ann. **346** (2010), no. 4, 969–989. 38

[42] Adrian Ioana, *W^*-superrigidity for Bernoulli actions of property (T) groups*, J. Amer. Math. Soc. **24** (2011), no. 4, 1175–1226. 52

[43] Adrian Ioana, Jesse Peterson, and Sorin Popa, *Amalgamated free products of weakly rigid factors and calculation of their symmetry groups*, Acta Math. **200** (2008), no. 1, 85–153. 19

[44] Barry E. Johnson, *Perturbations of Banach algebras*, Proc. London Math. Soc. (3) **34** (1977), no. 3, 439–458. 2, 3, 21

[45] ———, *A counterexample in the perturbation theory of C^*-algebras*, Canad. Math. Bull. **25** (1982), no. 3, 311–316. 2

[46] ———, *Approximately multiplicative maps between Banach algebras*, J. London Math. Soc. (2) **37** (1988), no. 2, 294–316. 3

[47] Vaughan F. R. Jones, *Index for subfactors*, Invent. Math. **72** (1983), no. 1, 1–25. 13, 14, 28

[48] Vaughan F. R. Jones and Viakalathur S. Sunder, *Introduction to subfactors*, London Mathematical Society Lecture Note Series, vol. 234, Cambridge University Press, Cambridge, 1997. 13

[49] Richard V. Kadison, *On the othogonalization of operator representations*, Amer. J. Math. **77** (1955), 600–620. 5

[50] ———, *Derivations of operator algebras*, Ann. of Math. (2) **83** (1966), 280–293. 10

[51] Richard V. Kadison and Daniel Kastler, *Perturbations of von Neumann algebras. I. Stability of type*, Amer. J. Math. **94** (1972), 38–54. 1, 3, 7, 11, 24, 37, 41, 43

[52] Richard V. Kadison and John R. Ringrose, *Derivations and automorphisms of operator algebras*, Comm. Math. Phys. **4** (1967), 32–63. 19

[53] ———, *Fundamentals of the theory of operator algebras. Vol. II*, Graduate Studies in Mathematics, vol. 16, American Mathematical Society, Providence, RI, 1997, Advanced theory, Corrected reprint of the 1986 original. 15, 18

[54] Daniel Každan, *On ε-representations*, Israel J. Math. **43** (1982), no. 4, 315–323. 3

[55] Mahmood Khoshkam, *On the unitary equivalence of close C^*-algebras*, Michigan Math. J. **31** (1984), no. 3, 331–338. 2

[56] ———, *Perturbations of C^*-algebras and K-theory*, J. Operator Theory **12** (1984), no. 1, 89–99. 14

[57] Eberhard Kirchberg, *The derivation problem and the similarity problem are equivalent*, J. Operator Theory **36** (1996), no. 1, 59–62. 12

[58] Dusa McDuff, *Central sequences and the hyperfinite factor*, Proc. London Math. Soc. (3) **21** (1970), 443–461. 41

[59] Igor Mineyev, *Straightening and bounded cohomology of hyperbolic groups*, Geom. Funct. Anal. **11** (2001), no. 4, 807–839. 56

[60] Nicolas Monod, *Continuous bounded cohomology of locally compact groups*, Lecture Notes in Mathematics, vol. 1758, Springer-Verlag, Berlin, 2001. 20

[61] ———, *On the bounded cohomology of semi-simple groups, S-arithmetic groups and products*, J. Reine Angew. Math. **640** (2010), 167–202. 2, 20, 21

[62] ———, *Personal communication*, 2011. 20

[63] Nicolas Monod and Yehuda Shalom, *Cocycle superrigidity and bounded cohomology for negatively curved spaces*, J. Differential Geom. **67** (2004), no. 3, 395–455. 2, 20

[64] Gerard J. Murphy, *C^*-algebras and operator theory*, Academic Press Inc., Boston, MA, 1990. 14

[65] Frank Murray and John von Neumann, *On rings of operators*, Ann. of Math. (2) **37** (1936), no. 1, 116–229. 18

[66] ———, *On rings of operators. IV*, Ann. of Math. (2) **44** (1943), 716–808. 14, 39

[67] Masahiro Nakamura, *On the direct product of finite factors*, Tôhoku Math. J. (2) **6** (1954), 205–207. 41, 50

[68] Masahiro Nakamura and Zirô Takeda, *On some elementary properties of the crossed products of von Neumann algebras*, Proc. Japan Acad. **34** (1958), 489–494. 18

[69] Narutaka Ozawa, *Solid von Neumann algebras*, Acta Math. **192** (2004), no. 1, 111–117. 37

[70] Narutaka Ozawa and Sorin Popa, *On a class of* II_1 *factors with at most one Cartan subalgebra*, Ann. of Math. (2) **172** (2010), no. 1, 713–749. 3, 6, 34, 38, 46

[71] ———, *On a class of* II_1 *factors with at most one Cartan subalgebra, II*, Amer. J. Math. **132** (2010), no. 3, 841–866. 6, 35, 38

[72] Robert Pallu de La Barrière, *Isomorphisme des* **-algèbres faiblement fermées d'opérateurs*, C. R. Acad. Sci. Paris **234** (1952), 795–797. 40

[73] John Phillips, *Perturbations of type I von Neumann algebras*, Pacific J. Math. **52** (1974), 505–511. 2

[74] John Phillips and Iain Raeburn, *Perturbations of AF-algebras*, Canad. J. Math. **31** (1979), no. 5, 1012–1016. 2

[75] ———, *Perturbations of* C^**-algebras. II*, Proc. London Math. Soc. (3) **43** (1981), no. 1, 46–72. 2

[76] Gilles Pisier, *The similarity degree of an operator algebra*, St. Petersburg Math. J. **10** (1999), no. 1, 103–146. 5, 12

[77] ———, *Remarks on the similarity degree of an operator algebra*, Internat. J. Math. **12** (2001), no. 4, 403–414. 12

[78] Florin Pop, Allan M. Sinclair, and Roger R. Smith, *Norming* C^**-algebras by* C^**-subalgebras*, J. Funct. Anal. **175** (2000), no. 1, 168–196. 9

[79] Sorin Popa, *On a problem of R. V. Kadison on maximal abelian* $*$*-subalgebras in factors*, Invent. Math. **65** (1981/82), no. 2, 269–281. 5, 9, 28, 37

[80] ———, *Maximal injective subalgebras in factors associated with free groups*, Adv. in Math. **50** (1983), no. 1, 27–48. 39

[81] ———, *Notes on Cartan subalgebras in type* II_1 *factors*, Math. Scand. **57** (1985), no. 1, 171–188. 53

[82] ———, *On a class of type* II_1 *factors with Betti numbers invariants*, Ann. of Math. (2) **163** (2006), no. 3, 809–899. 29, 34

[83] ———, *Strong rigidity of* II_1 *factors arising from malleable actions of w-rigid groups. I*, Invent. Math. **165** (2006), no. 2, 369–408. 2, 51

[84] ———, *Strong rigidity of* II_1 *factors arising from malleable actions of w-rigid groups. II*, Invent. Math. **165** (2006), no. 2, 409–451. 2, 51

[85] ———, *On Ozawa's property for free group factors*, Int. Math. Res. Not. IMRN (2007), no. 11, Art. ID rnm036, 10. 2, 52

[86] Sorin Popa and Dimitri Shlyakhtenko, *Cartan subalgebras and bimodule decompositions of* II_1 *factors*, Math. Scand. **92** (2003), no. 1, 93–102. 29

[87] Sorin Popa and Stefaan Vaes, *Unique Cartan decomposition for* II_1 *factors arising from arbitrary actions of free groups*, arXiv:1111.6951, 2011. 6, 35

[88] ———, *Unique Cartan decomposition for* II_1 *factors arising from arbitrary actions of hyperbolic groups*, arXiv:1201.2824, 2012. 6, 35, 56

[89] Iain Raeburn and Joseph L. Taylor, *Hochschild cohomology and perturbations of Banach algebras*, J. Funct. Anal. **25** (1977), no. 3, 258–266. 2, 3, 21

[90] Jean Roydor, *A non-commutative Amir-Cambern theorem for von-Neumann algebras*, arXiv.1108.1970v1, 2011. 11

[91] Klaus Schmidt, *Asymptotically invariant sequences and an action of* $\mathrm{SL}(2, \mathbf{Z})$ *on the* 2*-sphere*, Israel J. Math. **37** (1980), no. 3, 193–208. 46

[92] ———, *Amenability, Kazhdan's property T, strong ergodicity and invariant means for ergodic group-actions*, Ergodic Theory Dynamical Systems **1** (1981), no. 2, 223–236. 46

[93] Allan M. Sinclair and Roger R. Smith, *Hochschild cohomology of von Neumann algebras*, London Mathematical Society Lecture Note Series, vol. 203, Cambridge University Press, Cambridge, 1995. 10

[94] ———, *Hochschild cohomology for von Neumann algebras with Cartan subalgebras*, Amer. J. Math. **120** (1998), no. 5, 1043–1057. 9, 10

[95] ———, *Cartan subalgebras of finite von Neumann algebras*, Math. Scand. **85** (1999), no. 1, 105–120. 9

[96] ———, *Finite von Neumann algebras and masas*, London Mathematical Society Lecture Note Series, vol. 351, Cambridge University Press, Cambridge, 2008. 15, 27, 28, 36

[97] Thomas Sinclair, *Strong solidity of group factors from lattices in* $\mathrm{SO}(n,1)$ *and* $\mathrm{SU}(n,1)$, J. Funct. Anal. **260** (2011), no. 11, 3209–3221. 38

[98] Christian F. Skau, *Finite subalgebras of a von Neumann algebra*, J. Funct. Anal. **25** (1977), no. 3, 211–235. 13

[99] An Speelman and Stefaan Vaes, *A class of* II_1 *factors with many non-conjugate Cartan subalgebras*, Adv. in Math. **231** (2012), 2224–2251, arXiv:1107.1356. 35

[100] Colin E. Sutherland, *Cohomology and extensions of von Neumann algebras. II*, Publ. Res. Inst. Math. Sci. **16** (1980), no. 1, 134–174. 17

[101] Masamichi Takesaki, *Theory of operator algebras. III*, Encyclopaedia of Mathematical Sciences, vol. 127, Springer-Verlag, Berlin, 2003, Operator Algebras and Non-commutative Geometry, 8. 39

[102] Stefaan Vaes, *W*-superrigidity for Bernoulli actions and wreath product group von Neumann algebras*, Lecture Notes from VNG2011, Institut Henri Poincaré, Paris, 2011. 45

[103] Dan-Virgil Voiculescu, *The analogues of entropy and of Fisher's information measure in free probability theory. III. The absence of Cartan subalgebras*, Geom. Funct. Anal. **6** (1996), no. 1, 172–199. 38

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