RESEARCH STATEMENT

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1. Introduction

Von Neumann algebras arose in the work of Murray and von Neumann [14] as the symmetries of a quantum physical system. The state space of an isolated physical system can be modeled by a complex Hilbert space $\mathcal{H}$ and an observable of the state space by a bounded linear self-adjoint operator on $\mathcal{H}$. In theory, all that can be measured about the system is captured by the observable.

A von Neumann algebra is a subalgebra of the bounded linear operators on $\mathcal{H}$ ($B(\mathcal{H})$) that is unital (contains the identity operator on $\mathcal{H}$), closed under taking adjoints, and closed in strong operator topology (SOT). Von Neumann [25] established the fundamental double commutant theorem, one version of which states that a unital adjoint-closed subalgebra A of $B(\mathcal{H})$ is a von Neumann algebra if and only if $A = A'' = (A')'$, where A' denotes the set of all x in $B(\mathcal{H})$ that commute with A. This ensures that if S is any adjoint-closed set in $B(\mathcal{H})$, then $S' = S'''$. Taking $S = T = T^*$ gave von Neumann the connection between von Neumann algebras and quantum physics.

Murray and von Neumann realized that any von Neumann algebra could be built as a direct integral (a generalization of direct sums) out of *factors*, von Neumann algebras with only scalar multiples of the identity operator in their center. Many structural problems about von Neumann algebras then reduce to the factor case. Factors come in types I, II, and III.

My own area of interest is in type II factors, which are divided into types II_1 and II_∞. One can think of a II_1 factor M as an infinite dimensional analog of the $n \times n$ matrices, as they come equipped with a finite tracial state τ with the property that $\tau(x^*x) = 0$ implies $x = 0$ for all x in M. We can use τ to define an inner product $\langle x, y\rangle_\tau = \tau(xy^*)$ for x, $y \in M$. Completing M with respect to $\langle \cdot, \cdot \rangle_\tau$ yields a Hilbert space denoted by $L^2(M)$, and M is faithfully represented on $L^2(M)$ via left multiplication.

A von Neumann subalgebra N of a factor M which is itself a factor will be referred to as a *subfactor*. Vaughan Jones defined an index theory for subfactors of type II_1 factors, from which he deduced an invariant for knots, the Jones polynomial [11]. If $N \subseteq M \subseteq B(L^2(M))$, the index $[M : N]$ of N in M is the dimension of $L^2(M)$ as a left Hilbert N-module. If M is a II_1 factor and G is a contable discrete group with a proper, outer action θ on M, then if H is a subgroup of G, $[M \rtimes_\theta G : M \rtimes_\theta H] = [G : H]$. Hence the index in this case is always a positive integer, but the theory for general II_1 factors is far richer. It is Jones' remarkable rigidity result [11] that the index takes values only in the set $\{4\cos^2(\frac{\pi}{n}) : n \geq 3\} \cup [4, \infty]$, and that all such values are realized for subfactors of the hyperfinite II_1 factor.

The first application from the examination of finite index subfactors was Jones' discovery of the polynomial that now bears his name [11], which is a knot invariant. Also, the study of finite index subfactors has provided new solutions to the Yang-Baxter equations in statistical mechanics and new integrable lattice models [3], [13], building on constructions by Bisch and

Jones [1]. Finite index subfactors can also be used to generate topological quantum field theories [12], [6], [15].

If M is a II_1 factor and A is a masa, let $\mathcal{N}_M(A)$ be the group of normalizing unitaries of A in M, i.e. the set of all unitaries $u \in M$ such that $uAu^* = A$. The masa A is said to be

(1) Singular if $(\mathcal{N}_M(A))'' = A$;
(2) Semi-regular if $(\mathcal{N}_M(A))''$ is a subfactor of M;
(3) Regular if $(\mathcal{N}_M(A))'' = M$.

The notions of singularity, regularity, and semi-regularity for a masa A of a given factor M were first introduced by Dixmier [4], who showed that all three types of masas exist in the hyperfinite II_1 factor. These definitions can be extended to arbitrary subalgebras of M.

Recently, Popa and Ozawa [16] have obtained a new proof of Voiculescu's amazing theorem that $L(\mathbb{F}_n)$ has no regular masa [24] using intertwining techniques with partial isometries. In particular, they show that the normalizing algebra $\mathcal{N}_{L(\mathbb{F}_n)}(B)''$ of any diffuse, abelian subalgebra B of $L(\mathbb{F}_n)$ is hyperfinite. As $L(\mathbb{F}_n)$ itself is not hyperfinite, this implies Voiculescu's result, which reveals that not every II_1 factor can be obtained via the crossed product construction with a commutative von Neumann algebra. Hence, the study of normalizers of subalgebras gives powerful structural insight into the containing algebra.

2. Results

2.1. Normalizers and Tensor Products.

As a consequence of Popa's intertwining lemma [18], I showed in work with Sinclair, Smith, and White [22] that all singular masas in M have the weak asymptotic homomorphism property (WAHP) in M. In order to define this property, let ξ denote the identity of M, considered as an element of $L^2(M)$. Then for a subalgebra B of M, we denote by $L^2(B)$ the 2-norm closure of $B\xi$ in $L^2(M)$ and e_B the orthogonal projection onto $L^2(B)$. The unique normal, faithful, τ-preserving conditional expectation $\mathbb{E}_B$ from M to B is defined by $\mathbb{E}_B(x)\,\xi = e_B(x\xi)$. Following [20], B has the WAHP in M if

Definition 2.1. $\forall \varepsilon > 0$, $\forall x_1, \ldots, x_n, y_1, \ldots, y_m \in M$, $\exists$ unitary $u \in B$ with

$$\|\mathbb{E}_B(x_i u y_j) - \mathbb{E}_B(x_i)\, u \mathbb{E}_B(y_j)\|_2 < \varepsilon,$$

for all $1 \le i \le n$ and $1 \le j \le m$.

Since (WAHP)$\Rightarrow$(singular) is trivial, we immediately obtain

Theorem 2.2. *A masa A in a II_1 factor M is singular if and only if it has the WAHP.*

Given subalgebras B_i of II_1 factors M_i, $i = 1, 2$, it is easy to check that if B_1 and B_2 have the WAHP in M_1 and M_2, respectively, then $B_1 \overline{\otimes} B_2$ has the WAHP in $M_1 \overline{\otimes} M_2$, where $\overline{\otimes}$ denotes the von Neumann algebra tensor product. The following corollary is then an immediate consequence of Theorem 2.2:

Corollary 2.3. *If M_1 and M_2 are II_1 factors and A_1 and A_2 are singular masas in M_1 and M_2, respectively, then $A_1 \overline{\otimes} A_2$ is a singular masa in $M_1 \overline{\otimes} M_2$.*

Chifan [2] extended Corollary 2.3 to all masas in II_1 factors, proving that

$$\mathcal{N}_{M_1}(A_1)'' \overline{\otimes} \mathcal{N}_{M_2}(A_2)'' = \mathcal{N}_{M_1 \overline{\otimes} M_2}(A_1 \overline{\otimes} A_2)''.$$

For general unital subalgebras of II_1 factors, this result need not hold. To recapture results for tensor products, it is necessary to expand the class of normalizing operators to include partial isometries. Define the one-sided groupoid normalizers of B in M as

$$\mathcal{OG}_M(B) = \{v \in M : v^*v, vv^* \in \mathcal{P}(B) \text{ and } vBv^* \subseteq Bvv^*\}.$$

Call v a groupoid normalizer (denoted by $\mathcal{G}_M(B)$) if also $v^*Bv \subseteq B$. If B is a masa or a finite index subfactor, then every $v \in \mathcal{G}_M(B)$ will be of the form $v = up$ (this the content of Dye's Theorem [5]) for $u \in \mathcal{N}_M(B)$ and $p \in \mathcal{P}(B)$. Hence, we will obtain that $\mathcal{N}_M(B)'' = \mathcal{G}_M(B)''$ in these cases. For infinite index subfactors, it is still true that $v = up$ except now we can only expect $u \in \mathcal{ON}_M(B)$, and so

$$\mathcal{OG}_M(B)'' = \mathcal{ON}_M(B)''.$$

The following is the main result from [7]

Theorem 2.4. *Let B_1, B_2 be unital von Neumann subalgebras of M_1 and M_2 such that $B_i' \cap M_i \subseteq B_i$, i=1,2. Then*

$$\mathcal{OG}_{M_1}(B_1)''\overline{\otimes}\mathcal{OG}_{M_2}(B_2)'' = \mathcal{OG}_{M_1\overline{\otimes}M_2}(B_1\overline{\otimes}B_2)''$$

and

$$\mathcal{G}_{M_1}(B_1)''\overline{\otimes}\mathcal{G}_{M_2}(B_2)'' = \mathcal{G}_{M_1\overline{\otimes}M_2}(B_1\overline{\otimes}B_2)''.$$

This implies that the equality between the algebras generated by unitary normalizers holds for all von Neumann subalgebras of a II_1 factor satisfying Dye's theorem. We also obtained an explicit decomposition result for groupoid normalizers of a tensor product:

Corollary 2.5. *Let B_1, B_2 be as in Theorem 2.4. Suppose that $v \in \mathcal{G}_{B_1\overline{\otimes}B_2}(M_1\overline{\otimes}M_2)$ and let $\varepsilon > 0$. Then there are $x_1, \dots, x_n \in B_1\overline{\otimes}B_2$, $w_{1,1}, \dots, w_{1,n} \in \mathcal{G}_{M_1}(B_1)$ and $w_{2,1}, \dots, w_{2,n} \in \mathcal{G}_{M_2}(B_2)$ such that*

(1) $\|x_j\| \le 1$ for all $1 \le j \le n$.

(2) $\left\|v - \sum_{j=1}^n x_j(w_{1,j} \otimes w_{2,j})\right\|_2 < \varepsilon$.

2.2. **Strong Singularity.** Sinclair and Smith [21] introduced a metric condition for masas, α-strong singularity for $\alpha > 0$, as the existence of a number α with the property

$$\alpha\|u - \mathbb{E}_A(u)\|_2 \le \sup_{x\in M, \|x\|\le 1} \|\mathbb{E}_A(x) - \mathbb{E}_{uAu^*}(x)\|_2 \tag{2.1}$$

for all unitaries $u \in M$. If A is 1-strongly singular, we say that A is simply strongly singular. There is no difficulty in extending the definition to arbitrary subalgebras B of M, and in particular to subfactors N of M.

It was noted earlier that (WAHP)$\Rightarrow$(singularity), but actually (WAHP) $\Rightarrow$(strong singularity)$\Rightarrow$(singularity). By considering a Pimsner-Popa basis, in joint work with Grossman [8], we could show directly that

Proposition 2.6. *No proper finite index subfactor of a II_1 factor M can have the WAHP.*

Therefore, other techniques must be used when passing from singularity to α-strong singularity. Calculations can be simplified if the dimension of the higher relative commutant $N' \cap \langle M, e_N\rangle$ is small. In [8], we obtained the following

Theorem 2.7. *Suppose $N \subseteq M$ is an inclusion of II_1 factors with N singular in M and $N' \cap \langle M, e_N \rangle$ two-dimensional. Then for all unitaries $u \in M$,*

$$\sqrt{\frac{[M:N]-2}{[M:N]-1}}\|u - \mathbb{E}_N(u)\|_2 \leq \sup_{x \in M, \|x\| \leq 1} \|\mathbb{E}_N(x) - \mathbb{E}_{uNu^*}(x)\|_2$$

if $[M:N] < \infty$, so that N is $\sqrt{\frac{[M:N]-2}{[M:N]-1}}$-strongly singular in M. If $[M:N] = \infty$, then N is strongly singular in M.

The property $N' \cap \langle M, e_N \rangle \cong \mathbb{C} \oplus \mathbb{C}$ is referred to as 2-supertransitivity (or merely 2-transitivity) and is satisfied by all subfactors of noninteger index less than 4. For larger index values, we can take M to be the crossed product of a II_1 factor P by an outer action of $G = S_n$, and N the crossed product of P by $H \cong S_{n-1}$ a subgroup that fixes single element. One can also take $G = S_\infty$, the group of all permutations on $\mathbb{N}$ which fix all but finitely may elements, and H to be any subgroup fixing a single element as an example with infinite index.

Without assumptions on the higher relative commutants, we could show via a result of Popa, Sinclair, and Smith [19] that

Theorem 2.8. *Every singular subfactor M of a II_1 factor is α-strongly singular with $\alpha = \dfrac{1}{13}$.*

The existence of a global constant perhaps suggests the possibility of improvement. In fact, this is not the case. While the situation remains ambiguous for infinite index subfactors, in the finite index setting, Grossman and I showed in [8] that the ultimate improvement is impossible:

Theorem 2.9. *There exists a finite index singular subfactor N of the hyperfinite II_1 factor $\mathcal{R}$ such that N is no more than $\sqrt{2(\sqrt{2}-1)}$ strongly singular in $\mathcal{R}$.*

The hyperfinite II_1 factor may be obtained from the crossed product by L_∞ with an amenable group, and is the unique (up to isomorphism) II_1 factor that is the strong closure of finite dimensional algebras, from whence the term "hyperfinite" arises. The example is a GHJ subfactor obtained from D_5 at the trivalent vertex. Planar algebra techniques are used to compute that the expectation of the unitary in question is zero, while the angles between the subfactor and its unitary conjugate bound the norm difference of the conditional expectation below 1. While lacking the tidiness of the masa case, the possibilities for finite index subfactors yield a much more interesting theory. The supremum of all numbers α appearing in equation (2.1) is a new, nontrivial numerical invariant for singular subfactors under inner conjugacy.

3. Further Directions of Research

I am interested in pursuing the computation of strong singularity constants, in particular:

Question 3.1. What are the possible strong singularity constants for GHJ subfactors or for $N = P \rtimes S_{n-1} \subseteq P \rtimes S_n = M$?

The GHJ construction provides a pair of singular subfactors with the same index. The difficulty lies in showing there is a unitary in the ambient factor that conjugates one to the other. Although it is easier to find natural unitaries to test in the group-subgroup case, all calculations up to this point have yielded $\alpha = 1$. I would also like to know,

Question 3.2. Does there exist an analog of Theorem 2.7 for $N' \cap M_1$ merely finite-dimensional?

I think the best that can be hoped for is a constant that tends to one as the index goes to infinity. From discussions with Ionut Chifan, I suspect a bound dependent upon the trace of a nonzero minimal projection p subordinate to $e_N^{\perp}$ in $N' \cap M_1$ of the form $\alpha \geq \sqrt{1 - \frac{1}{\mathrm{Tr}(p)}}$ to be possible by employing averaging techniques, but this does not have any nice limiting properties. Since every known example of a strongly singular subfactor has the WAHP, we can ask,

Question 3.3. Does $\alpha = 1$ characterize the WAHP?

A potential candidate for study is the subfactor inclusion described at the end of [23]. We have $M = L(G) \supset L(H) = N$ where $G = \mathbb{F}_\infty \rtimes (\mathbb{Z} \rtimes \mathbb{Z}_2)$ and H is the subgroup generated by the copies of $\mathbb{F}_\infty$ and $\mathbb{Z}_2$. Then $[M : N] = [G : H] = \infty$ and the minimal trace of a nonzero subprojection of $e_N^{\perp}$ is two. Therefore N is singular in M but does not have the WAHP as a consequence of Popa's intertwining lemma.

Aside from the computation of strong singularity constants, a problem I have thought a little about is,

Question 3.4. Is there a masa in every separable II_1 factor with Pukanszky invariant equal to $\{\infty\}$?

It seems it might be possible to adapt Popa's construction of a singular masa in any separable II_1 factor [17] to produce such a masa.

Leaving the type II realm, there has been progress in understanding the structure of preduals type III hyperfinite factors. These factors are divided into subtypes III_λ where $0 \leq \lambda \leq 1$. An open question is

Question 3.5. What are the complete or Banach isomorphism classes for hyperfinite type III_0 factor preduals?

By results of Haagerup, Sukochev, and Rosenthal [10], there is a single complete isomorphism class for type III_λ factor preduals when $\lambda > 0$. It is a recent result of Haagerup and Musat [9] that there are uncountably many complete isomorphism classes for type III_0 preduals. The Banach isomorphism question appears to be quite difficult.

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