A REMARK ON THE SIMILARITY AND PERTURBATION PROBLEMS

JAN CAMERON, ERIK CHRISTENSEN, ALLAN M. SINCLAIR, ROGER R. SMITH, STUART WHITE, AND ALAN D. WIGGINS

Abstract. In this note we show that Kadison's similarity problem for C^*-algebras is equivalent to a problem in perturbation theory: must close C^*-algebras have close commutants?

Let $\mathcal{A}$ be a C^*-algebra. In 1955 Kadison asked whether every bounded homomorphism $\sigma : \mathcal{A} \to \mathbb{B}(\mathcal{H})$ is similar to a *-homomorphism [12], i.e. does there exist an invertible operator $S \in \mathbb{B}(\mathcal{H})$ such that $S^{-1}\sigma(\cdot)S$ is a *-homomorphism? This problem, which remains open, has positive answers in the following cases: for amenable algebras [2], for properly infinite von Neumann algebras and traceless C^*-algebras [10], in the presence of a cyclic vector [10], and for II_1 factors with property Γ [8]. It is equivalent to a number of important problems and properties:

- is every bounded homomorphism $\sigma : \mathcal{A} \to \mathbb{B}(\mathcal{H})$ completely bounded? ([10])
- the *derivation problem* [15] - given a faithful non-degenerate representation $\iota : \mathcal{A} \to \mathbb{B}(\mathcal{H})$, is every derivation of $\iota(\mathcal{A})$ into $\mathbb{B}(\mathcal{H})$ spatial?
- the *distance property* [5, 7, 15] - does there exist a constant $K > 0$ such that given any faithful non-degenerate representation $\iota : A \to \mathbb{B}(\mathcal{H})$

$$\frac{1}{2}\|\mathrm{ad}(T)|_{\iota(A)}\|_{cb} = d(T, \iota(\mathcal{A})') \leq K\|\mathrm{ad}(T)|_{\iota(A)}\| \tag{1}$$

  holds for all $T \in \mathbb{B}(\mathcal{H})$?[1] When this holds, $\mathcal{A}$ is said to have property D_K. Here, and throughout the paper, $\mathrm{ad}(T)$ denotes the spatial derivation $X \mapsto [T, X] = TX - XT$ on $\mathbb{B}(\mathcal{H})$.
- do all C^*-algebras have finite length?[2] ([17])
- are all von Neumann algebras hyperreflexive? (see [18, Theorem 10.3]).
- Diximer's invariant operator range problem from 1950 (see [18, Problem 10.4]).

In this note we add to the list of equivalent formulations by showing that the similarity problem is equivalent to an open question in the uniform perturbation theory of operator algebras.

The Kadison-Kastler metric on all C^*-subalgebras of $\mathbb{B}(\mathcal{H})$ is given by restricting the Hausdorff metric on subsets of $\mathbb{B}(\mathcal{H})$ to the unit balls. Specifically, for pairs of C^*-algebras

Date: June 26, 2012.

JC's research is partially supported by an AMS-Simons research travel grant.

RS is partially supported by NSF grant DMS-1101403.

SW is partially supported by EPSRC grant EP/I019227/1.

[1]The equality in (1) is Arveson's distance formula from [1] and is always valid.

[2]Pisier's characterisation of the similarity problem using his notion of *length* provides an intrinsic formulation of the similarity property in terms of internal behaviour.

$\mathcal{A}, \mathcal{B} \subseteq \mathbb{B}(\mathcal{H})$,

$$d(\mathcal{A}, \mathcal{B}) = \max\left\{ \sup_{\substack{x\in\mathcal{A}\\ \|x\|\leq 1}} \inf_{\substack{y\in\mathcal{B}\\ \|y\|\leq 1}} \|x-y\|, \sup_{\substack{y\in\mathcal{B}\\ \|y\|\leq 1}} \inf_{\substack{x\in\mathcal{A}\\ \|y\|\leq 1}} \|y-x\| \right\}.$$

This notion was introduced in [13] which conjectures that sufficiently close von Neumann algebras must arise from a small unitary perturbation. Precisely, for all $\varepsilon > 0$, there should exist $\delta > 0$ such that, given von Neumann algebras $\mathcal{M}, \mathcal{N} \subseteq \mathbb{B}(\mathcal{H})$ with $d(\mathcal{M}, \mathcal{N}) < \delta$, there exists a unitary U on $\mathcal{H}$ with $\|U - I_{\mathcal{H}}\| < \varepsilon$ satisfying $U\mathcal{M}U^* = \mathcal{N}$.[3] This conjecture, if true, immediately implies that the operation of taking commutants of C^*-algebras is continuous with respect to the Kadison-Kastler metric.[4] Establishing that certain close operator algebras have close commutants, and a related property of near inclusions (see Remark 2 below) play a key role in our recent work on the Kadison-Kastler problem for crossed product factors [3], in EC's solution to the near inclusion problem for injective von Neumann algebras [6] and in work on transferring K-theoretic invariants between close algebras [14, 9, 16] (see Remark 3 below). Here we show that the continuity of the operation of taking commutants is equivalent to the similarity problem. In order to phrase this result for individual C^*-algebras we make the following definition.

Definition. Let $\mathcal{A}$ be a C^*-algebra. Say that *commutants are continuous at* $\mathcal{A}$ if, for $\varepsilon > 0$, there exists $\delta > 0$ such that given a faithful non-degenerate *-representation $\iota : \mathcal{A} \to \mathbb{B}(\mathcal{H})$ and a C^*-algebra $\mathcal{B} \subseteq \mathbb{B}(\mathcal{H})$ with $d(\iota(\mathcal{A}), \mathcal{B}) < \delta$, we have $d(\iota(\mathcal{A})', \mathcal{B}') < \varepsilon$.

Theorem. *The following statements are equivalent for a C^*-algebra $\mathcal{A}$:*

(I) $\mathcal{A}$ satisfies Kadison's similarity property;

(II) commutants are continuous at $\mathcal{A}$.

Further, the similarity problem has a positive answer for all C^*-algebras if and only if the operation of taking commutants on $\mathbb{B}(\mathcal{H})$ is continuous in the case when $\mathcal{H}$ is an infinite dimensional separable Hilbert space.

Proof of the Theorem

If the similarity problem has a positive answer for all C^*-algebras, then there exists a constant $K > 0$ such that all C^*-algebras have property D_K (by [15] and [7]). It is then easy to see that

$$d(\mathcal{A}', \mathcal{B}') \leq 4K d(\mathcal{A}, \mathcal{B}) \tag{2}$$

for all C^*-algebras $\mathcal{A}, \mathcal{B} \subset \mathbb{B}(\mathcal{H})$ (this dates back to [6], and can be found explicitly by combining Proposition 2.3 (iii) and Proposition 2.5 of [9]). The local version (I) $\Longrightarrow$ (II) of this implication is much more involved as we do not have the information that $\mathcal{B}$ also has property D_K required in the previous argument. Nevertheless, this implication was established in [9, Theorem 4.2] as a step towards showing that if $\mathcal{A}$ has the similarity property, then any C^*-algebra sufficiently close to $\mathcal{A}$ also has the similarity property.

[3]Due to the results of [4, 11], the analogous conjecture for C^*-algebras is that sufficiently close separable C^*-algebras acting on a separable Hilbert space should be spatially isomorphic; one cannot demand control on the distance from an implementing unitary to the identity in this context.

[4]Via a Kaplansky density argument, it suffices to consider the continuity of taking commutants on the set of von Neumann subalgebras of $\mathbb{B}(\mathcal{H})$.

We now turn to the reverse implication (II) $\Longrightarrow$ (I). For $X, R \in \mathbb{B}(\mathcal{H})$, we use the identity $R^m X - X R^m = R^{m-1}(RX - XR) + (R^{m-1}X - XR^{m-1})R$ to establish by induction the estimates

$$\|[R^m, X]\| \leq m\|R\|^{m-1}\|[R, X]\|, \qquad m \in \mathbb{N}. \tag{3}$$

Suppose that commutants are continuous at $\mathcal{A}$, but that there exists no constant $K > 0$ such that $\mathcal{A}$ has property D_K. Then we may find a sequence $(\iota_n)_n$ of faithful, non-degenerate representations $\iota : \mathcal{A} \to \mathbb{B}(\mathcal{H}_n)$ and operators $T_n \in \mathbb{B}(\mathcal{H}_n)$ with

$$d(T_n, \iota_n(\mathcal{A})') > n\|\mathrm{ad}(T_n)|_{\iota_n(\mathcal{A})'}\|, \qquad n \geq 1.$$

For each n, choose $\widetilde{S_n} \in \iota_n(\mathcal{A})'$ satisfying $\|\widetilde{S_n} - T_n\| = d(T_n, \iota_n(A)')$. Write $S_n = T_n - \widetilde{S_n}$ so that

$$\|S_n\| > n\|\mathrm{ad}(S_n)|_{\iota_n(\mathcal{A})'}\|, \qquad n \geq 1. \tag{4}$$

Scaling in this inequality allows us to assume that $\|S_n\| = 1$ while retaining the validity of (4). Since $d(S_n, \iota_n(\mathcal{A})') = 1$, either $d(\Re(S_n), \iota_n(\mathcal{A})') \geq 1/2$ or $d(\Im(S_n), \iota_n(\mathcal{A})') \geq 1/2$ for each n, while the equation $\|\mathrm{ad}(S_n)|_{\iota_n(\mathcal{A})}\| = \|\mathrm{ad}(S_n^*)|_{\iota_n(\mathcal{A})}\|$ gives the limiting behaviour $\|\mathrm{ad}(\Re(S_n))|_{\iota_n(\mathcal{A})}\|, \|\mathrm{ad}(\Im(S_n))|_{\iota_n(\mathcal{A})}\| \to 0$ as $n \to \infty$. Thus we obtain a sequence of self-adjoint contractions $R_n \in \mathbb{B}(\mathcal{H}_n)$ such that $d(R_n, \iota_n(\mathcal{A})') \geq 1/2$ for $n \geq 1$ and $\lim_{n\to\infty} \|\mathrm{ad}(R_n)|_{\iota_n(\mathcal{A})}\| = 0$. By approximating $f(t) = \sqrt{1 - t^2}$ uniformly by polynomials on $[-1, 1]$ and using the estimate (3), we see that the unitary operators

$$U_n^{\pm} = R_n \pm i\sqrt{1 - R_n^2}$$

satisfy $\lim_{n\to\infty} \|\mathrm{ad}(U_n^{\pm})|_{\iota_n(\mathcal{A})}\| = 0$. Since $R_n = (U_n^+ + U_n^-)/2$ for each n, either $d(U_n^+, \iota_n(\mathcal{A})') \geq 1/2$ or $d(U_n^-, \iota_n(\mathcal{A})') \geq 1/2$. Appropriate choices of signs give unitaries $U_n \in \mathbb{B}(\mathcal{H}_n)$ with

$$d(U_n, \iota_n(\mathcal{A})') \geq 1/2, \qquad n \geq 1, \tag{5}$$

while $\lim_{n\to\infty} \|\mathrm{ad}(U_n)|_{\iota_n(\mathcal{A})}\| = 0$.

Now consider the faithful non-degenerate representations $\iota_n \oplus \iota_n : \mathcal{A} \to \mathbb{M}_2(\mathbb{B}(\mathcal{H}_n))$ so that

$$(\iota_n \oplus \iota_n)(\mathcal{A}) = \left\{ \begin{pmatrix} \iota_n(a) & 0 \\ 0 & \iota_n(a) \end{pmatrix} : a \in \mathcal{A} \right\}.$$

Define C^*-algebras $\mathcal{B}_n$ by

$$\mathcal{B}_n = \left\{ \begin{pmatrix} \iota_n(a) & 0 \\ 0 & U_n\iota_n(a)U_n^* \end{pmatrix} : a \in \mathcal{A} \right\} \subseteq \mathbb{M}_2(\mathbb{B}(\mathcal{H}_n)), \qquad n \geq 1,$$

so that $d((\iota_n \oplus \iota_n)(\mathcal{A}), \mathcal{B}_n) \leq \|\mathrm{ad}(U_n)|_{\iota_n(\mathcal{A})}\| \to 0$. Clearly $W_n = \begin{pmatrix} 0 & I_{\mathcal{H}_n} \\ 0 & 0 \end{pmatrix} \in (\iota_n \oplus \iota_n)(\mathcal{A})'$ so, by hypothesis, $d(W_n, \mathcal{B}_n') \to 0$. Choose $\widetilde{W_n} \in \mathcal{B}_n'$ with $\|W_n - \widetilde{W_n}\| = d(W_n, \mathcal{B}_n')$ and let X_n be the $(1, 2)$-entry of $\widetilde{W_n}$. Then $\|X_n - I_{\mathcal{H}_n}\| \to 0$ and

$$\iota_n(a)X_n = X_n U_n \iota_n(a) U_n^*, \quad a \in \mathcal{A}.$$

Thus $X_n U_n \in \iota_n(\mathcal{A})'$ so that $d(U_n, \iota_n(\mathcal{A})') \to 0$, contradicting (5). This completes the proof of (II) $\Longrightarrow$ (I).

Finally, note that Kadison's similarity problem is a equivalent to the existence of a constant $K > 0$ such that

$$\frac{1}{2}\|\mathrm{ad}(T)|_{\mathcal{A}}\|_{cb} \leq K\|\mathrm{ad}(T)|_{\mathcal{A}}\|, \quad T \in \mathbb{B}(\mathcal{H}),$$

whenever $\mathcal{H}$ is a *separable* Hilbert space and $\mathcal{A} \subseteq \mathbb{B}(\mathcal{H})$ is a C^*-algebra. This is established by a standard argument (cf the last two paragraphs of the proof of [3, Proposition 2.1.5]) which shows that this last condition implies the same condition without the separability assumption. Consequently, the proof of (II) $\Longrightarrow$ (I) shows that if the operation of taking commutants is continuous on $\mathbb{B}(\mathcal{H})$ for $\mathcal{H}$ separable and infinite dimensional, then Kadison's similarity property holds.

Remarks

1. If the operation of taking commutants is globally continuous, then there is a universal constant $K > 0$ such that all C^*-algebras have property D_K and hence (2) holds. In particular taking commutants is Lipschitz. In the local situation, the proof of (I) $\Longrightarrow$ (II) given in [9, Theorem 4.2] shows that if $\mathcal{A} \subseteq \mathbb{B}(\mathcal{H})$ is a C^*-algebra which has finite similarity length ℓ and length constant K, then there exists a constant $C'_{\ell,K}$ such that

$$d(\mathcal{A}', \mathcal{B}') \leq C'_{\ell,K} d(\mathcal{A}, \mathcal{B})$$

for all C^*-algebras $\mathcal{B}$ on $\mathcal{H}$ with $d(\mathcal{A}, \mathcal{B})$ sufficiently small. Since the metric d has diameter 1, it follows that there is a constant $C_{\ell,K}$ such that

$$d(\mathcal{A}', \mathcal{B}') \leq C_{\ell,K} d(\mathcal{A}, \mathcal{B})$$

for all C^*-algebras $\mathcal{B}$ on $\mathcal{H}$. This says that if $\mathcal{A}$ has the similarity property, then taking commutants is locally a Lipschitz operation near $\mathcal{A}$.

2. The similarity property is also characterised by the property that near inclusions of algebras give rise to near inclusions of their commutants. For C^*-algebras $\mathcal{A}, \mathcal{B} \subseteq \mathbb{B}(\mathcal{H})$ and $\gamma \geq 0$, write $\mathcal{A} \subseteq_\gamma \mathcal{B}$ if, given $x \in \mathcal{A}$, there exists $y \in \mathcal{B}$ with $\|x - y\| \leq \gamma \|x\|$. It is natural to ask whether, given $\varepsilon > 0$, there exists $\delta > 0$ such that the near inclusion $\mathcal{A} \subseteq_\delta \mathcal{B}$ implies that $\mathcal{B}' \subseteq_\varepsilon \mathcal{A}'$.[5] Indeed the connection between the similarity problem and questions of close commutants originates in [6] where a positive answer is given when $\mathcal{A}$ has property D_K. This question is also equivalent to the similarity property, and the local version at $\mathcal{A}$ is equivalent to the similarity property for $\mathcal{A}$. The global statement follows immediately, as the ability to take commutants of all near inclusions in this way implies that taking commutants is continuous. For the local statement, note that the proof of the theorem only uses the continuity of commutants for pairs $(\mathcal{A}, U\mathcal{A}U^*)$, and this follows from being able to take commutants of near inclusions of $\mathcal{A}$.

3. It is natural to consider a completely bounded version of the Kadison-Kastler metric: $d_{cb}(\mathcal{A}, \mathcal{B}) = \sup_n d(\mathcal{A} \otimes \mathbb{M}_n, B \otimes \mathbb{M}_n)$.[6] When $d_{cb}(\mathcal{A}, \mathcal{B})$ is sufficiently small, one can show that $\mathcal{A}$ and $\mathcal{B}$ have the same K-theoretic invariants used in the classification programme, [14, 9]. Moreover in this case $\mathcal{A}$ and $\mathcal{B}$ also have isomorphic Cuntz semigroups [16]. Thus it is of interest to learn when $d(\cdot, \cdot)$ and $d_{cb}(\cdot, \cdot)$ are equivalent metrics: this too is a reformulation of the similarity problem.[7] That the similarity problem implies the equivalence of these

[5] One can also formulate this question locally at $\mathcal{A}$ by asking that this hold under all faithful non-degenerate representations of $\mathcal{A}$.

[6] Similarly define $\mathcal{A} \subset_{cb,\gamma} \mathcal{B}$ if and only if $\mathcal{A} \otimes \mathbb{M}_n \subset_\gamma \mathcal{B} \otimes \mathbb{M}_n$ for all $n \geq 1$.

[7] A local version of this statement also holds: $\mathcal{A}$ has the similarity property if and only if $d_{cb}(\mathcal{A}, \cdot)$ is equivalent to $d(\mathcal{A}, \cdot)$.

metrics is noted at the end of [9, Section 4]. The converse follows as Arveson's distance formula shows that $\mathcal{A} \subset_{cb,\gamma} \mathcal{B} \implies \mathcal{B}' \subset_{cb,\gamma} \mathcal{A}'$ (see [3, Proposition 2.2.3]) and so

$$d(\mathcal{A}', \mathcal{B}') \leq d_{cb}(\mathcal{A}', \mathcal{B}') \leq 2d_{cb}(\mathcal{A}, \mathcal{B}).$$

Acknowledgements. When SW presented the work of [3] in Vanderbilt University in January 2012, Dietmar Bisch asked whether it was necessary to use the similarity property in order to take commutants of close C^*-algebras. This directly sparked the present paper and we are grateful to Dietmar for bringing this question to our attention.

References

[1] William B. Arveson, *Interpolation problems in nest algebras*, J. Funct. Anal. **20** (1975), 208–233.
[2] John W. Bunce, *The similarity problem for representations of C^*-algebras*, Proc. Amer. Math. Soc. **81** (1981), no. 3, 409–414.
[3] Jan Cameron, Erik Christensen, Allan M. Sinclair, Roger R. Smith, Stuart White, and Alan D. Wiggins, *Kadison-Kastler stability of II_1-factors*, Preprint, 2012.
[4] Man-Duen Choi and Erik Christensen, *Completely order isomorphic and close C^*-algebras need not be *-isomorphic*, Bull. London Math. Soc. **15** (1983), no. 6, 604–610.
[5] Erik Christensen, *Extension of derivations*, J. Funct. Anal. **27** (1978), no. 2, 234–247.
[6] ______, *Near inclusions of C^*-algebras*, Acta Math. **144** (1980), no. 3-4, 249–265.
[7] ______, *Extensions of derivations. II*, Math. Scand. **50** (1982), no. 1, 111–122.
[8] ______, *Similarities of* II_1 *factors with property* Γ, J. Operator Theory **15** (1986), no. 2, 281–288.
[9] Erik Christensen, Allan M. Sinclair, Roger R. Smith, and Stuart White, *Perturbations of C^*-algebraic invariants*, Geom. Funct. Anal. **20** (2010), no. 2, 368–397.
[10] Uffe Haagerup, *Solution of the similarity problem for cyclic representations of C^*-algebras*, Ann. of Math. (2) **118** (1983), no. 2, 215–240.
[11] Barry E. Johnson, *A counterexample in the perturbation theory of C^*-algebras*, Canad. Math. Bull. **25** (1982), no. 3, 311–316.
[12] Richard V. Kadison, *On the othogonalization of operator representations*, Amer. J. Math. **77** (1955), 600–620.
[13] Richard V. Kadison and Daniel Kastler, *Perturbations of von Neumann algebras. I. Stability of type*, Amer. J. Math. **94** (1972), 38–54.
[14] Mahmood Khoshkam, *Perturbations of C^*-algebras and K-theory*, J. Operator Theory **12** (1984), no. 1, 89–99.
[15] Eberhard Kirchberg, *The derivation problem and the similarity problem are equivalent*, J. Operator Theory **36** (1996), no. 1, 59–62.
[16] Fransesc Perera, Andrew S. Toms, Stuart White, and Wilhelm Winter, *The Cuntz semigroup of close C^*-algebras*, Manuscript in preparation, 2012.
[17] Gilles Pisier, *The similarity degree of an operator algebra*, Algebra i Analiz **10** (1998), no. 1, 132–186.
[18] ______, *Similarity problems and completely bounded maps*, expanded ed., Lecture Notes in Mathematics, vol. 1618, Springer-Verlag, Berlin, 2001, Includes the solution to "The Halmos problem".

Jan Cameron, Department of Mathematics, Vassar College, Poughkeepsie, New York, U.S.A.
E-mail address: jacameron@vassar.edu

Erik Christensen, Institute for Mathematiske Fag, University of Copenhagen, Copenhagen, Denmark.
E-mail address: echris@math.ku.dk

Allan M. Sinclair, School of Mathematics, University of Edinburgh, JCMB, King's Buildings, Mayfield Road, Edinburgh, EH9 3JZ, Scotland.
E-mail address: a.sinclair@ed.ac.uk

Roger R. Smith, Department of Mathematics, Texas A&M University, College Station, Texas, 77843, U.S.A.
E-mail address: rsmith@math.tamu.edu

Stuart White, School of Mathematics and Statistics, University of Glasgow, University Gardens, Glasgow Q12 8QW, Scotland.
E-mail address: stuart.white@glasgow.ac.uk

Alan D. Wiggins, Department of Mathematics and Statistics. University of Michigan-Dearborn, 4901 Evergreen Road, Dearborn, Michigan, 48126, USA.
E-mail address: adwiggin@umd.umich.edu