STRONG SINGULARITY OF SINGULAR MASAS IN II_1 FACTORS

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Abstract

A singular masa A in a II_1 factor N is defined by the property that any unitary $w \in N$ for which $A = wAw^*$ must lie in A. A strongly singular masa A is one that satisfies the inequality

$$\|\mathbb{E}_A - \mathbb{E}_{wAw^*}\|_{\infty,2} \geq \|w - \mathbb{E}_A(w)\|_2$$

for all unitaries $w \in N$, where $\mathbb{E}_A$ is the conditional expectation of N onto A, and $\|\cdot\|_{\infty,2}$ is defined for bounded maps $\phi : N \to N$ by $\sup\{\|\phi(x)\|_2 : x \in N,\ \|x\| \leq 1\}$. Strong singularity easily implies singularity, and the main result of this paper shows the reverse implication.

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1 Introduction

In [13], the first two authors introduced the concept of a strongly singular maximal abelian self–adjoint subalgebra (masa) in a II_1 factor N. For a bounded map $\phi : M \to N$ between any two finite von Neumann algebras with specified traces, the $\|\cdot\|_{\infty,2}$–norm is defined by

$$\|\phi\|_{\infty,2} = \sup\{\|\phi(x)\|_2 : x \in M,\ \|x\| \leq 1\}. \tag{1.1}$$

A masa $A \subseteq N$ is then said to be *strongly singular* when the inequality

$$\|\mathbb{E}_A - \mathbb{E}_{wAw^*}\|_{\infty,2} \geq \|w - \mathbb{E}_A(w)\|_2 \tag{1.2}$$

holds for all unitaries $w \in N$, where the notation $\mathbb{E}_B$ indicates the unique trace preserving conditional expectation onto a von Neumann subalgebra B. Any unitary which normalizes A is forced, by this relation, to lie in A, and so A is singular, as defined in [3]. The original purpose for introducing strong singularity was to have a metric condition which would imply singularity, and which would be easy to verify in a wide range of cases (see [13, 14] and the work of the third author, [15], on Tauer masas in the hyperfinite II_1 factor R). Subsequently, [11], it was shown that every singular masa in a separable II_1 factor (where this terminology indicates norm–separability of the predual N_*) satisfies the following weaker inequality, analogous to (1.2):

$$90\|\mathbb{E}_A - \mathbb{E}_{wAw^*}\|_{\infty,2} \geq \|w - \mathbb{E}_A(w)\|_2 \tag{1.3}$$

holds for all unitaries $w \in N$. This clearly suggested that every singular masa should be strongly singular, and our objective in this paper is to prove this result.

In 1983 Sorin Popa introduced the δ–invariant for a masa in a II_1 factor, [7]. This was the first attempt to define a metric based invariant for a masa, which he used to show that there is an abundance of singular masas in separable II_1 factors, [7]. Subsequently Popa, [8], showed that a masa in a separable II_1 factor is singular if, and only if, it has δ–invariant 1. An invariant $\alpha(A)$ for a masa A in a II_1 factor, based on unitary perturbations of A, was defined by the first two authors in [13] with strong singularity corresponding to $\alpha(A) = 1$. In that paper the masa A was shown to be singular if $\alpha(A) > 0$. Theorem 2.3 implies that a masa in a separable II_1 factor is singular if, and only if, it is strongly singular. This is the analogous result to Popa's δ–invariant one with unitaries in N replacing his nilpotent partial isometries, whose domains and ranges are orthogonal projections in the masa. For a masa A in a separable II_1 factor the results in [8] show that $\delta(A)$ is either 0 or 1, and Theorem 2.3 implies that $\alpha(A)$ also takes only these two values (see [13] for the definition of $\alpha(A)$).

The main result of the paper is Theorem 2.3, the proof of which is given in the lemmas that precede it. These in turn are based on results of Sorin Popa, [10, Thm. 2.1, Cor. 2.3], which have their origin in [9]. We also give two applications of our results. One shows the singularity of tensor products of singular masas (Corollary 2.4), while the other shows that singular masas can usually be studied in the setting of separable algebras (Theorem 2.5).

Much of the work in this paper was accomplished at the Spring Institute on Noncommutative Geometry and Operator Algebras, held May 9–20 2005 at Vanderbilt University. The lecture series presented by Sorin Popa at this conference provided the basis for our results below. It is a pleasure to express our gratitude to the organizers and to the NSF for providing financial support during the conference.

2 Main Results

Our first lemma is essentially contained in [10, Corollary 2.3]. The proof will amount to identifying the subalgebras to which this corollary is applied.

Lemma 2.1. *Let A be a masa in a II_1 factor N and let $e, f \in A$ be nonzero projections with the property that no nonzero partial isometry $w \in N$ satisfies the conditions*

$$ww^*, ww^* \in A,\ ww^* \leq e,\ w^*w \leq f,\ \text{and } w^*Aw = Aw^*w. \tag{2.1}$$

If $\varepsilon > 0$ and $x_1, \ldots, x_k \in N$ are given, then there exists a unitary $u \in A$ such that

$$\|\mathbb{E}_A(fx_ieux_j^*f)\|_2 < \varepsilon \tag{2.2}$$

for $1 \leq i, j \leq k$.

Proof. Define two subalgebras $B_0 = Ae$ and $B = Af$ of N. The hypothesis implies the negation of the fourth condition for B_0 and B in [10, Theorem 2.1] and so [10, Corollary 2.3] can be applied. Thus, given $a_1, \ldots, a_k \in N$ and $\varepsilon > 0$, there exists a unitary $u_1 \in B_0$ such that

$$\|\mathbb{E}_B(a_iu_1a_j^*)\|_2 < \varepsilon, \quad 1 \leq i, j \leq k. \tag{2.3}$$

The result follows from this by taking $a_i = fx_ie$, $1 \leq i \leq k$, replacing u_1 by the unitary $u = u_1 + (1-e) \in A$, and replacing $\mathbb{E}_B$ in (2.3) by $\mathbb{E}_A$. Note that these two conditional expectations agree on fNf. □

Below, we will use the notation $\mathcal{U}(M)$ for the unitary group of any von Neumann algebra M. We will also need the well known fact that if $x \in M$ and B is a von Neumann subalgebra, then $\mathbb{E}_{B'\cap M}(x)$ is the unique element of minimal norm in the $\|\cdot\|_2$–closure of conv $\{uxu^* \colon u \in \mathcal{U}(B)\}$. We do not have an exact reference for this, but it is implicit in [1].

Lemma 2.2. *Let A be a singular masa in a II_1 factor N. If $x_1, \ldots, x_k \in N$ and $\varepsilon > 0$ are given, then there is a unitary $u \in A$ such that*

$$\|\mathbb{E}_A(x_iux_j^*) - \mathbb{E}_A(x_i)u\mathbb{E}_A(x_j^*)\|_2 < \varepsilon \tag{2.4}$$

for $1 \leq i, j \leq k$.

Proof. If $x, y \in N$ and $u \in A$, then

$$\mathbb{E}_A(xuy^*) - \mathbb{E}_A(x)u\mathbb{E}_A(y^*) = \mathbb{E}_A((x - \mathbb{E}_A(x))u\mathbb{E}_A(y - \mathbb{E}_A(y))^*) \tag{2.5}$$

by the module properties of $\mathbb{E}_A$. Thus (2.4) follows if we can establish that

$$\|\mathbb{E}_A(x_iux_j^*)\|_2 < \varepsilon, \quad 1 \leq i, j \leq k, \tag{2.6}$$

when the x_i's also satisfy $\mathbb{E}_A(x_i) = 0$ for $1 \le i \le k$. We assume this extra condition, and prove (2.6). By scaling, there is no loss of generality in assuming $\|x_i\| \le 1$ for $1 \le i \le k$.

Let $\delta = \varepsilon/4$. In the separable case, [7] gives a finite dimensional abelian subalgebra $A_1 \subseteq A$ with minimal projections $e_1, \ldots, e_n$ such that

$$\|\mathbb{E}_{A_1' \cap N}(x_i) - \mathbb{E}_A(x_i)\|_2 < \delta \tag{2.7}$$

for $1 \le i \le k$. The assumption that $\mathbb{E}_A(x_i) = 0$ allows us to rewrite (2.7) as

$$\|\mathbb{E}_{A_1' \cap N}(x_i)\|_2 = \|\mathbb{E}_A(x_i) - \mathbb{E}_{A_1' \cap N}(x_i)\|_2 < \delta \tag{2.8}$$

for $1 \le i \le k$, leading to

$$\|\sum_{m=1}^{n} e_m x_i e_m\|_2 = \|\mathbb{E}_{A_1' \cap N}(x_i)\|_2 < \delta \tag{2.9}$$

since $\sum_{m=1}^{n} e_m x_i e_m = \mathbb{E}_{A_1' \cap N}(x_i)$. Any partial isometry $v \in N$ satisfying $vAv^* = Avv^*$ has the form pu for a projection $p \in A$ and a normalizing unitary $u \in N$, [4, 5]. The singularity of A then shows that $v \in A$, making it impossible to satisfy the two inequalities $vv^* \le e_m$ and $v^*v \le (1-e_m)$ simultaneously unless $v = 0$. Thus no nonzero partial isometry $v \in N$ satisfies $vv^* \le e_m$, $v^*v \le (1 - e_m)$, and $vAv^* = Avv^*$. The hypothesis of Lemma 2.1 is satisfied, and applying this result with ε replaced by δ/n gives unitaries $u_m \in A$ such that

$$\|\mathbb{E}_A((1 - e_m)x_i e_m u_m x_j^*(1 - e_m))\|_2 < \delta/n \tag{2.10}$$

for $1 \le m \le n$ and $1 \le i, j \le k$. Define a unitary $u \in A$ by $u = \sum_{m=1}^{n} u_m e_m$, and let $y_i = \sum_{m=1}^{n}(1 - e_m)x_i e_m$, for $1 \le i \le k$. We have

$$x_i - y_i = x_i - \sum_{m=1}^{n}(1 - e_m)x_i e_m = \sum_{m=1}^{n} e_m x_i e_m = \mathbb{E}_{A_1' \cap N}(x_i) \tag{2.11}$$

for $1 \le i \le k$. The inequalities

$$\|x_i - y_i\|_2 < \delta, \quad \|x_i - y_i\| \le \|x_i\| \le 1, \quad \|y_i\| \le 2, \tag{2.12}$$

for $1 \le i \le k$, follow immediately from (2.8) and (2.11).

If we apply $\mathbb{E}_A$ to the identity

$$x_i u x_j^* = (x_i - y_i)u x_j^* + y_i u(x_j^* - y_j^*) + y_i u y_j^*, \tag{2.13}$$

then (2.12) gives

$$\begin{aligned} \|\mathbb{E}_A(x_i u x_j^*)\|_2 &\le \|x_i - y_i\|_2 + \|y_i\|\|x_j - y_j\|_2 + \|\mathbb{E}_A(y_i u y_j^*)\|_2 \\ &< 3\delta + \|\mathbb{E}_A(y_i u y_j^*)\|_2, \end{aligned} \tag{2.14}$$

and we estimate the last term. The identity

$$y_iuy_j^* = \sum_{m,s=1}^{n} (1-e_m)x_ie_mue_sx_j^*(1-e_s) = \sum_{m=1}^{n}(1-e_m)x_ie_mu_mx_j^*(1-e_m) \tag{2.15}$$

holds because each e_s commutes with u and $e_me_s = 0$ for $m \neq s$. The last sum has n terms, so the inequalities

$$\|\mathbb{E}_A(y_iuy_j^*)\|_2 < \delta, \quad 1 \leq i, j \leq k, \tag{2.16}$$

are immediate from (2.10). Together (2.14) and (2.16) yield

$$\|\mathbb{E}_A(x_iux_j^*)\|_2 < 3\delta + \delta = \varepsilon, \quad 1 \leq i, j \leq k, \tag{2.17}$$

as required.

In the general case, we obtain A_1 and (2.7) as follows. Since A is a masa,

$$\mathbb{E}_{A' \cap N}(x_i) = \mathbb{E}_A(x_i) = 0, \quad 1 \leq i \leq k. \tag{2.18}$$

Now $\mathbb{E}_{A'\cap N}(x_i)$ is the element of minimal $\|\cdot\|_2$–norm in the $\|\cdot\|_2$–closed convex hull of $\{wx_iw^* \colon w \in \mathcal{U}(A)\}$, so we may select a finite number of unitaries $w_1, \ldots, w_r \in A$ such that each set $\Omega_i = \operatorname{conv}\{w_jx_iw_j^* \colon 1 \leq j \leq r\}$, $1 \leq i \leq k$, contains an element whose $\|\cdot\|_2$–norm is less than δ. The spectral theorem allows us to make the further assumption that each w_j has finite spectrum, whereupon these unitaries generate a finite dimensional subalgebra $A_1 \subseteq A$. Then, for $1 \leq i \leq k$, $\mathbb{E}_{A_1'\cap N}(x_i)$ is the element of smallest norm in the $\|\cdot\|_2$–closed convex hull of $\{wx_iw^* \colon w \in \mathcal{U}(A_1)\}$, and since this set contains Ω_i, we see that (2.7) is valid in general. $\square$

In [13], a masa A in a II_1 factor N was defined to have the *asymptotic homomorphism property* (*AHP*) if there exists a unitary $v \in A$ such that

$$\lim_{|n|\to\infty} \|\mathbb{E}_A(xv^ny) - \mathbb{E}_A(x)v^n\mathbb{E}_A(y)\|_2 = 0 \tag{2.19}$$

for all $x, y \in N$. In that paper it was shown that strong singularity is a consequence of this property. Subsequently it was observed in [12, Lemma 2.1] that a weaker property, which we will call the *weak asymptotic homomorphism property*, (*WAHP*), suffices to imply strong singularity: given $\varepsilon > 0$ and $x_1, \ldots, x_k, y_1, \ldots, y_k \in N$, there exists a unitary $u \in A$ such that

$$\|\mathbb{E}_A(x_iuy_j) - \mathbb{E}_A(x_i)u\mathbb{E}_A(y_j)\|_2 < \varepsilon \tag{2.20}$$

for $1 \leq i, j \leq k$. Since the *WAHP* is a consequence of applying Lemma 2.2 to the set of elements $x_1, \ldots, x_k, y_1^*, \ldots, y_k^* \in N$, we immediately obtain the main result of the paper from these remarks:

Theorem 2.3. *Let A be a singular masa in a* II_1 *factor N. Then A has the WAHP and is strongly singular.*

The following observation on the tensor product of masas may be known to some experts, but we have not found a reference.

Corollary 2.4. *For $i = 1, 2$, let $A_i \subseteq N_i$ be masas in* II_1 *factors. If A_1 and A_2 are both singular, then $A_1 \overline{\otimes} A_2$ is also a singular masa in $N_1 \overline{\otimes} N_2$.*

Proof. Lemma 2.2 and the remarks preceding Theorem 2.3 show that singularity and the *WAHP* are equivalent for masas in II_1 factors. By Tomita's commutant theorem, $A_1 \overline{\otimes} A_2$ is a masa in $N_1 \overline{\otimes} N_2$, and it is straightforward to verify that the *WAHP* carries over to tensor products (see [16, Proposition 1.4.27]), since it suffices to check this property on a $\|\cdot\|_2$–norm dense set of elements, in this case the span of $\{x \otimes y \colon x \in N_1,\ y \in N_2\}$. □

As an application of these results, we end by showing that the study of singular masas can, in many instances, be reduced to the separable case. The techniques of the proof have their origin in [2, Section 7].

Theorem 2.5. *Let N be a* II_1 *factor with a singular masa A and let M_0 be a separable von Neumann subalgebra of N. Then there exists a separable subfactor M such that $M_0 \subseteq M \subseteq N$ and $M \cap A$ is a singular masa in M.*

Proof. We will construct M as the weak closure of the union of an increasing sequence $M_0 \subseteq M_1 \subseteq M_2 \subseteq \ldots$ of separable von Neumann subalgebras, chosen by induction. These will have an increasing sequence of abelian subalgebras $B_k \subseteq M_k$ with certain properties.

For a von Neumann algebra $Q \subseteq N$ and for any $x \in N$, $K_Q(x)$ will denote the set $\mathrm{conv}\,\{uxu^* \colon u \in \mathcal{U}(Q)\}$, and the $\|\cdot\|$– and $\|\cdot\|_2$–closures will be denoted $K_Q^n(x)$ and $K_Q^w(x)$ respectively. The inclusions $K_Q(x) \subseteq K_Q^n(x) \subseteq K_Q^w(x)$ are immediate. The induction hypothesis is: each M_k is separable, $M_k \subseteq M_{k+1}$, and for a fixed sequence $\{y_{k,r}\}_{r=1}^{\infty}$ in the unit ball of M_k which is $\|\cdot\|_2$–dense in the $\|\cdot\|_2$–closure of this ball,

(i) $\mathbb{E}_A(y_{k,r}) \in B_{k+1} \cap K_{B_{k+1}}^w(y_{k,r})$ for $r \geq 1$, where $B_{k+1} = M_{k+1} \cap A$;

(ii) $K_{M_{k+1}}^n(y_{k,r}) \cap \mathbb{C}1$ is nonempty for $r \geq 1$;

(iii) given $\varepsilon > 0$, $r \geq 1$ and a projection $p \in B_k$, there exists $u \in \mathcal{U}(B_{k+1})$ such that

$$\|\mathbb{E}_A((1-p)y_{k,s}puy_{k,t}^*(1-p))\|_2 < \varepsilon$$

for all $1 \leq s, t \leq r$.

We first show that such a sequence of algebras leads to the desired conclusion. Let M and B be respectively the weak closures of the unions of the M_k's and B_k's. Since $K^n_{M_k}(x) \subseteq K^n_M(x)$ for all $x \in M$ and $k \geq 1$, condition (ii) and a simple approximation argument show that $K^w_M(x)$ contains a scalar operator for all x in the unit ball of M, and thus for all $x \in M$ by scaling. Since $K^w_M(z) = \{z\}$ for any central element $z \in M$, this shows that M is a factor, separable by construction.

Now consider $x \in M$; scaling allows us to assume without loss of generality that $\|x\| \leq 1$. Condition (i) shows that $\mathbb{E}_A(y_{k,r}) \in K^w_B(y_{k,r})$ for $k, r \geq 1$, and an approximation argument then shows that $\mathbb{E}_A(x) \in K^w_B(x) \subseteq K^w_A(x)$. Since $\mathbb{E}_A(x)$ is the element of minimal $\|\cdot\|_2$–norm in $K^w_A(x)$, it also has this property in $K^w_B(x)$. But this minimal element is $\mathbb{E}_{B' \cap M}(x)$, showing that $\mathbb{E}_{B'\cap M}(x) = \mathbb{E}_A(x)$. If we further suppose that $x \in B' \cap M$, then $x = \mathbb{E}_A(x)$. Thus $B' \cap M \subseteq A$ and is abelian. Condition (i) also shows that $\mathbb{E}_A(x) \in B$, and so $B' \cap M \subseteq B$. Since B is abelian, we have equality, proving that B is a masa in M. We can now conclude that $\mathbb{E}_A(y) = \mathbb{E}_B(y)$ for all $y \in M$, and that $B = M \cap A$.

Another $\|\cdot\|_2$–approximation argument, starting from (iii), gives

$$\inf \{ \max_{1 \leq i,j \leq r} \|\mathbb{E}_B((1-p)x_i p u x_j^*(1-p))\|_2 \colon u \in \mathcal{U}(B)\} = 0 \tag{2.21}$$

for an arbitrary finite set of elements $x_1, \ldots, x_r \in M$, $\|x_i\| \leq 1$, and any projection $p \in B$, noting that $\mathbb{E}_A$ and $\mathbb{E}_B$ agree on M. By scaling, it is clear that this equation holds generally without the restriction $\|x_i\| \leq 1$. We have now established the conclusion of Lemma 2.1 for B from which singularity of B follows, as in the proof of Lemma 2.2. It remains to construct the appropriate subalgebras M_k.

To begin the induction, let $B_0 = M_0 \cap A$, and suppose that $B_k \subseteq M_k$ have been constructed. Consider a fixed sequence $\{y_{k,r}\}_{r=1}^\infty$ in the unit ball of M_k, $\|\cdot\|_2$–dense in the $\|\cdot\|_2$–closure of this ball. The Dixmier approximation theorem, [6, Theorem 8.3.5], allows us to obtain a countable number of unitaries, generating a separable subalgebra $Q_0 \subseteq N$, so that $K^n_{Q_0}(y_{k,r}) \cap \mathbb{C}1$ is nonempty for $r \geq 1$. Let $\{p_m\}_{m=1}^\infty$ be a sequence which is $\|\cdot\|_2$–dense in the set of all projections in B_k. The singularity of A ensures that the hypothesis of Lemma 2.1 is met when $e = p_m$ and $f = 1 - p_m$. Thus, for integers $m, r, s \geq 1$, there is a unitary $u_{m,r,s} \in A$ such that

$$\|\mathbb{E}_A((1-p_m)y_{k,i}p_m u_{m,r,s} y_{k,j}^*(1-p_m))\|_2 < 1/s \tag{2.22}$$

for $1 \leq i, j \leq r$. These unitaries generate a separable von Neumann algebra $Q_1 \subseteq A$, and an approximation argument establishes (iii) provided that $Q_1 \subseteq B_{k+1}$.

Since $\mathbb{E}_A(y_{k,r})$ is the minimal $\|\cdot\|_2$–norm element in $K^w_A(y_{k,r})$, we may find a countable number of unitaries generating a separable subalgebra $Q_2 \subseteq A$ so that $\mathbb{E}_A(y_{k,r}) \in K^w_{Q_2}(y_{k,r})$ for $r \geq 1$. The proof is completed by letting M_{k+1} be the separable von Neumann algebra generated by M_k, $\mathbb{E}_A(M_k)$, Q_0, Q_1 and Q_2. The subspaces $\mathbb{E}_A(M_k)$ and Q_2 are included to ensure that (i) is satisfied, (ii) holds by the choice of Q_0, and Q_1 guarantees the validity of (iii). $\square$

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