Math 412/512 Assignment 1

Due Wednesday, January 19

1) Given two vectors $\vec{v} = \langle v_1, v_2, v_3\rangle$ and $\vec{w}\langle w_1, w_2, w_3\rangle$ in $\mathbb{R}^3$, recall that their *cross-product* is given by

$$\vec{v} \times \vec{w} = \langle v_2w_3 - v_3w_2, v_3w_1 - v_1w_3, v_1w_2 - v_2w_1\rangle.$$

Note that " $\times$ " is a (binary) operation.

a) Is " $\times$ " commutative? Justify your answer.

b) Is " $\times$ " associative? Justify your answer.

c) Is there an identity for " $\times$ "? If so, exhibit the identity. If not, show why there can be no identity.

2) (See p. 30 in the text) If A and B are two subsets of a larger set D, their *symmetric difference* is the set $A + B$ defined by

$$A + B = (A\backslash B) \cup (B\backslash A) = (A \cap B^c) \cup (B \cap A^c)$$

where the notation A^c refers to all elements in D that are NOT in A, called the complement of A.

The *power set* of D is the set P_D of all subsets of D. Consider " $+$ " as a binary operation on P_D.

a) Exhibit the identity element of $\langle P_D, +\rangle$.

b) Given $A \subset D$, exhibit the inverse of A in $\langle P_D, +\rangle$. Thus, $\langle P_D, +\rangle$ is a group.

c) Let D be the three-element set $D = \{a, b, c\}$. List the elements of P_D. Do not forget the empty set and the whole set D. Then write the operation table for $\langle P_D, +\rangle$.

3) There are two possible distinct (up to isomorphism) group structures on a four element set $\{a, b, c, d\}$. Determine them, i.e., construct the operation tables. (*Hint:* both groups are commutative.)

4) Show that if $\langle G, \cdot \rangle$ is a group and $x, y \in G$, $x \cdot y = y \cdot x$, then $(x \cdot y)^n = x^n \cdot y^n$.

5) Prove that, with the operation of multiplication, $G = \{x + y\sqrt{17} \mid x, y \in \mathbb{Q}\} \backslash \{0\}$ is a group. You may assume associativity is inherited from $\langle \mathbb{R} \backslash \{0\}, \cdot \rangle$.