**Math 412/512 Assignment 2**

**Due Wednesday, January 26**

**1)** Consider a four element set $S = \{a, b, c, d\}$. Consider the following operation tables on $S$, both of which induce a group structure on $S$.

| $\cdot$ | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| $a$ | a | b | c | d |
| $b$ | b | d | a | c |
| $c$ | c | a | d | b |
| $d$ | d | c | b | a |

| $*$ | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| $a$ | a | b | c | d |
| $b$ | b | c | d | a |
| $c$ | c | d | a | b |
| $d$ | d | a | b | c |

Define $\phi : \langle S, \cdot \rangle \to \langle S, * \rangle$ by

$$\phi(a) = a, \quad \phi(b) = b, \quad \phi(c) = d, \quad \phi(d) = c.$$

Show that, for all elements $n$ and $m$ in $\{b, c, d\}$, $\phi(n \cdot m) = \phi(n) * \phi(m)$.

**2)** Determine that the groups $\mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ are not isomorphic. *Hint:* argue by contradiction.

**3)** (See Chapter 9, Section E, #4 in the text) Show that $\langle \mathbb{R} \backslash \{0\}, \cdot \rangle$ and $\langle \mathbb{R}, + \rangle$ are not isomorphic. *Hint:* consider $-1 \in \langle \mathbb{R} \backslash \{0\}, \cdot \rangle$.

**4)** Let $S_\infty$ denote the set of all bijections from $\mathbb{N}$ to $\mathbb{N}$ that fix all but a finite number of elements, with function composition as the operation. So for example,

$$\phi(n) = \begin{cases} 1 & \text{if } n = 2 \\ 2 & \text{if } n = 1 \\ n & \text{if } n \geq 2 \end{cases}$$

determines an element $\phi \in S_\infty$ since only two elements of $\mathbb{N}$ are not fixed by $\phi$, but

$$\gamma(n) = \begin{cases} n + 1 & \text{if } n \text{ is odd} \\ n - 1 & \text{if } n \text{ is even} \end{cases}$$

does not determine an element of $S_\infty$ since $\gamma$ does not fix any elements of $\mathbb{N}$.

a) Show that $S_\infty$ is a group.

b) Let $T \subset S_\infty$ be the subset of all bijections $\phi$ with the property that $\phi(1) = 1$. Is $T$ is a subgroup of $S_\infty$?

**5)** (See Chapter 5, Section D) Let $\langle G, \cdot \rangle$ be a group.

a) Suppose $H$ and $K$ are subgroups of $\langle G, \cdot \rangle$. Show that $H \cap K$ is also a subgroup of $\langle G, \cdot \rangle$.

b) Is it always the case that if $H$ and $K$ are subgroups of $\langle G, \cdot \rangle$ then $H \cup K$ is a subgroup of $\langle G, \cdot \rangle$? Prove or give a counterexample.

c) Let $H$ be a subgroup of $\langle G, \cdot \rangle$ and for $x \in G$, let

$$xHx^{-1} = \{xax^{-1} : a \in H\}.$$

Define

$$K = \{x \in G : xHx^{-1} = H.\}$$

Show that $H \subseteq K$ and that $K$ is a subgroup of $\langle G, \cdot \rangle$. $K$ is called the *normalizer* of $H$ in $\langle G, \cdot \rangle$, and is usually denoted by $N_G(H)$.

d) If $\langle G, \cdot \rangle$ is abelian, determine the normalizer of all subgroups $H$ of $\langle G, \cdot \rangle$.

**6)** (see the previous question) A subgroup $H$ of $\langle G, \cdot \rangle$ is called *malnormal* if $N_G(H) = H$. Prove that $S_3$ has a proper malnormal subgroup.

**7)** Is the set of nonzero real numbers whose square is an irrational number a subgroup of $\langle \mathbb{R} \backslash \{0\}, \cdot \rangle$?