Math 412/512 Assignment 4

Due Wednesday, February 16

1) (See Chapter 9, Section H) Show that if G and H are groups, then $G \times H$ is isomorphic to $H \times G$.

2) Recall that D_3 is the group of symmetries of an equilateral triangle. The proof of Cayley's Theorem given in class yields a subgroup of S_6 that is isomorphic to D_3. After labeling the elements of D_3, explicitly produce the subgroup given by the proof, i.e., determine all the elements.

3) (See Chapter 10, Section E)

a) Suppose $g, h \in G$ and that $gh = hg$. Prove that $ord(gh)$ is a divisor of $lcm(ord(g), ord(h))$.

b) Let $\sigma = (127), \tau = (59368) \in S_9$. Calculate the order of $\sigma\tau$.

c) Give an example of a group G and elements $g, h \in G$ such that $ord(gh)$ is strictly greater than $lcm(ord(g), ord(h))$.

4) If $g \in G$, define the *conjugacy class* of g to be the subset of G consisting of all elements of the form hgh^{-1} for $h \in G$.

a) If $g \in \mathcal{Z}(G)$, determine the conjugacy class of g.

b) G is said to have the *infinite conjugacy class* (ICC) condition if the conjugacy class of any nonidentity element $g \in G$ has infinite cardinality. Show that S_∞, as defined on your second homework set, has the ICC condition. *Hint:* consider conjugation by transpositions.

c) Determine $\mathcal{Z}(S_\infty)$.

d) Extra credit: Does $SL_3(\mathbb{Z})$ have the ICC? Prove or disprove.

5) Let $G = GL_2(\mathbb{R})$ and $H \leq G$ consist of all $A \in G$ with $det(A) > 0$. Calculate $[G : H]$.

6) Let G be an infinite group and let $H \leq G \times G$ be the subgroup of all pairs of the form (g, g) with $g \in G$. Calculate $[G \times G : H]$.

7) (super bonus extra credit) If G is a group and the order of every element in G is finite, does this imply that G is a finite group? Prove or give a counterexample.