Math 412/512 Assignment 5

Due Friday, March 18

1) (See Chapter 14, Section B) Let $\phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $\phi((x, y)) = x + y$.

a) Show that ϕ is a homomorphism

b) Determine $ker(\phi)$.

c) Use the First Isomorphism Theorem to characterize $(\mathbb{R} \times \mathbb{R})/ker(\phi)$ as a familiar group.

2) (See Chapter 15, Section F and Chapter 16, Section D) Recall that $Inn(G) \triangleleft Aut(G)$ is the group of all inner automorphisms of a group G; that is, $\phi \in Inn(G)$ if there exists a $g \in G$ such that for all $k \in G$, $\phi(k) = gkg^{-1}$. Recall also that $Z(G) = \{g \in G : gk = kg \text{ for all} k \in G\}$.

a) Show that $Inn(G)$ is isomorphic to $G/Z(G)$.

b) Prove that if $G/Z(G)$ is cyclic, then G is abelian.

3) Provide an example that shows if $K \leq H \leq G$, $H \triangleleft G$, and $K \triangleleft H$, then K is not necessarily normal in G.

4) $H \leq G$ is called *characteristic* in G if for all $\phi \in Aut(G)$, $\phi(H) = H$.

a) Show that a characteristic subgroup is automatically normal.

b) Give an example of a nontrivial (i.e. neither the identity nor the whole group) characteristic subgroup of $\mathbb{Z}_6$. Be sure to prove that your example is, indeed, characteristic.

c) Show that if $K \leq H \leq G$, $H \triangleleft G$, and K is characteristic in H, then $K \triangleleft G$.

d) (Extra Credit) Show that every subgroup of a cyclic group is characteristic.

5) Determine the isomorphism class of $GL_n(\mathbb{R})/SL_n(\mathbb{R})$, i.e., find a familiar group that is isomorphic to $GL_n(\mathbb{R})/SL_n(\mathbb{R})$. *Hint:* First Isomorphism Theorem.

6) Let $H \leq G$, $[G : H] = \infty$. Show that the number (cardinality) of left cosets of H in G is equal to the number (cardinality) of right cosets of H in G.