Math 412/512 Assignment 6

Due Friday, March 25

1) Complete the proof, begun in class, that $\mathbb{Z}[x]$ is a ring by showing distributivity.

2) (See Chapter 17, Section E) In $M_2(\mathbb{C})$, let

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \mathbf{i} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$$\mathbf{j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ \mathbf{k} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

a) Verify that $\mathbf{ij} = \mathbf{k}$, $\mathbf{jk} = \mathbf{i}$, $\mathbf{ki} = \mathbf{j}$, and $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -I$.

b) Show that if a, b, c, and d are real numbers, then

$$S = \{aI + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \mid a, b, c, \text{ and } d \in \mathbb{R}\}$$

is a subring of $M_2(\mathbb{C})$.

c) Prove that there is a nontrivial ring structure on $\mathbb{R}^4$.

3) Show that if R is a ring, I is an ideal of R, and 1_R is the multiplicative identity of R, then $1_R \in I$ implies $I = R$.

4) The upper triangular matrices consist of all $T = (T_{i,j})_{i,j=1}^n \in M_n(\mathbb{R})$ such that $T_{i,j} = 0$ if $i > j$.

a) Exhibit an upper triangular matrix in $M_3(\mathbb{R})$.

b) Show that the upper triangular matrices in $M_3(\mathbb{R})$ are a subring of $M_3(\mathbb{R})$.

c) Are the upper triangular matrices in $M_3(\mathbb{R})$ an ideal in $M_3(\mathbb{R})$? Prove or disprove.

d) (Extra Credit) Prove that $M_n(\mathbb{R})$ admits no nontrivial proper ideals. You may receive some credit by demonstrating this fact for $M_3(\mathbb{R})$.

5) (See Chapter 18, Section H) Recall that $C_0(\mathbb{R})$ is the ring of all continuous function $f : \mathbb{R} \to \mathbb{R}$ with $\lim_{|x| \to \infty} f(x) = 0$. Fix a real number x and define

$$I_x = \{f \in C_0(\mathbb{R}) \mid f(x) = 0\}.$$

a) Show that, for any choice of x, I_x is an ideal in $C_0(\mathbb{R})$.

b) Prove that I_x is a maximal ideal.