**Math 412/512 Assignment 7**

**Due Wednesday, April 6**

**1)** For a unital ring $R$, let $R^\times$ denote the set of all units.

a) Prove that $\langle R^\times, \cdot\rangle$ is a group, where "$\cdot$" is the multiplication operation on $R$.

b) Determine the group isomorphism class of $\mathbb{Z}[x]^\times$.

c) If $R$ and $S$ are unital rings, determine $(R \times S)^\times$.

**2)** (Group ring) Let $G$ be a finite group. Recall from Cayley's Theorem that there exists an injective homomorphism $\phi : G \to S_n$ where $n = |G|$. Also note that for a matrix $T \in M_n(\mathbb{C})$, $T_{i,j}$ refers to the entry of $T$ in the $i$th row and $j$th column for $1 \leq i, j \leq n$.

a) Define $\psi_n : S_n \to GL_n(\mathbb{C})$ by

$$(\psi_n(\sigma))_{i,j} = \begin{cases} 1 & \text{if } \sigma(i) = j \\ 0 & \text{otherwise} \end{cases}$$

for all $\sigma \in S_n$. Prove that $\psi$ is well-defined (i.e. that $\psi(\sigma)$ is actually invertible for all $\sigma \in S_n$) and a homomorphism of groups.

b) Define the complex *Group ring* of $G$ to be the linear span of the image $\psi_n(\phi(G))$. Show that this is, indeed, a ring with unit. The notation is $\mathbb{C}[G]$.

c) Determine necessary and sufficient conditions on $G$ that guarantee $\mathbb{C}[G]$ is commutative.

**3)** a) Let $R$ be an integral domain and let $x, y \in R$. Show that if $x^2 = y^2$, then $x = y$ or $x = -y$.

b) Give an example of a ring $R$ where $x^2 = y^2$ does not necessarily imply either $x = y$ or $x = -y$ for $x, y \in R$. Be sure to prove that your example actually works.

c) (Extra Credit) Provide an example of a ring $R$ with no zero divisors where $x^2 = y^2$ does not necessarily imply either $x = y$ or $x = -y$ for $x, y \in R$.

Again, be sure to prove that your example is correct. Note that your answer for b) may provide you with such an example!

**4)** Show that every field is a Euclidean domain.

**5)** Prove that $\mathbb{Z}[x]$ is not a principal ideal domain. *Hint:* Consider $\{p \in \mathbb{Z}[x] \mid p \text{ has an even constant coefficient}\}$.