**Math 412/512 Assignment 1**

**Due Tuesday, September 20**

**1)** Use mathematical induction to complete parts a) and b).

a) (#50, Chapter 0) Let $S$ be a set with $n$ elements for some $n \in \mathbb{N}$. Prove that there are $2^n$ distinct subsets of $S$.

b) Show that $\displaystyle\sum_{i=1}^{n} i^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$.

c) Extra credit: do part b) without using induction!

**2)** Let $S, T$, and $U$ be sets and let $f : T \to U$, $g : S \to T$.

a) Prove that if $f \circ g$ is bijective, then $f$ is surjective and $g$ is injective.

b) If $f$ is injective and $g$ is surjective, does it then follow that $f \circ g$ must be bijective? Prove or give a counterexample.

c) Verify that if $S$ is finite, then any injection from $S$ to itself is also surjective.

d) Give an example of an injection $f : \mathbb{N} \to \mathbb{N}$ that is not surjective.

**3)** Let $S = \{n \in \mathbb{Z} : \text{either } 2|n \text{ or } 3|n\}$. Determine whether $S$ is a group under the operation of addition.

**4)** Recall that for a $2 \times 2$ matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{C})$,

$$\det(A) = ad - bc.$$

Prove or disprove that $S = \{A \in M_2(\mathbb{C}) : |\det(A)| = 1\}$ is a group with the operation of matrix multiplication (you may assume associativity).

**5)** (#26, Chapter 2) Let $G$ be a group. Show that if $g, h \in G$ and $(g \cdot h)^2 = g^2 \cdot h^2$, then $G$ is abelian (recall that for $k \in G$, $k^2$ is defined as $k \cdot k$).