Math 412/512 Assignment 2

Due Thursday, September 29

1) Let

$$G = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\} \subseteq M_3(\mathbb{R})$$

a) Check that G is a group with the operation of matrix multiplication.

b) Prove that

$$H = \left\{ \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid a \in \mathbb{R} \right\}$$

is a subgroup of G.

2) Let H be a subgroup of a group G and suppose G has group operation "$*$". For $x \in G$, let

$$xHx^{-1} = \{x * h * x^{-1} : h \in H\}.$$

Define

$$K = \{x \in G : xHx^{-1} = H.\}$$

a) Show that $H \subseteq K$.

b) Prove that K is a subgroup of G. K is called the *normalizer* of H in G, and is usually denoted by $N_G(H)$.

c) Let G be abelian and suppose $H \leq G$. Determine $N_G(H)$.

3) Consider S_n as the group of all bijections on $\{1, 2, \ldots, n\}$.

a) Let $T \subset S_n$ be the subset of all bijections ϕ with the property that $\phi(1) = 1$. Is T is a subgroup of S_n?

b) Now let $B \subset S_n$ be the subset of all bijections ψ with the property that $\psi(1) \neq 1$. Is B a subgroup of S_n?

c) (see the previous question) A subgroup H of a group G is called *malnormal* if $N_G(H) = H$. Prove that S_3 has a proper malnormal subgroup.

4) (#14, Chapter 3) Suppose that H is a proper subgroup of $\mathbb{Z}$ and H contains 18, 30, and 40. Determine H.

5) (#18, Chapter 3) a) Suppose H and K are subgroups of G. Show that $H \cap K$ is also a subgroup of G.

b) Is it always the case that if H and K are subgroups of G then $H \cup K$ is a subgroup of G? Prove or give a counterexample.